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# Three Families of Mitered Borromean Ring Sculptures

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**Abstract**

Artists have drawn inspiration from many mathematical structures, such as regular tilings, lattices, symmetry groups, regular polyhedra, knots, and links. A particularly well-known link goes by the name *Borromean rings*. It consists of three closed loops ("rings") that cannot be taken apart without cutting, but after removing any one of the rings, the other two can be separated without cutting. In this paper, we present three families of sculptures involving miter joints, inspired by the Borromean rings.

## 1 Introduction

Figure 1 (left) shows the famous *Borromean rings*, a link consisting of three closed loops, which cannot be taken apart without cutting. However, when a single loop is removed, the other two turn out to be unlinked (see Fig. 1, right). In the figure, the top left-hand ring (red) is completely above the top right-hand ring (green), which in turn is completely above the bottom ring (blue). Finally, the blue ring is completely above the red ring. No pair of rings is linked; nevertheless the three cannot be unlinked without cutting.

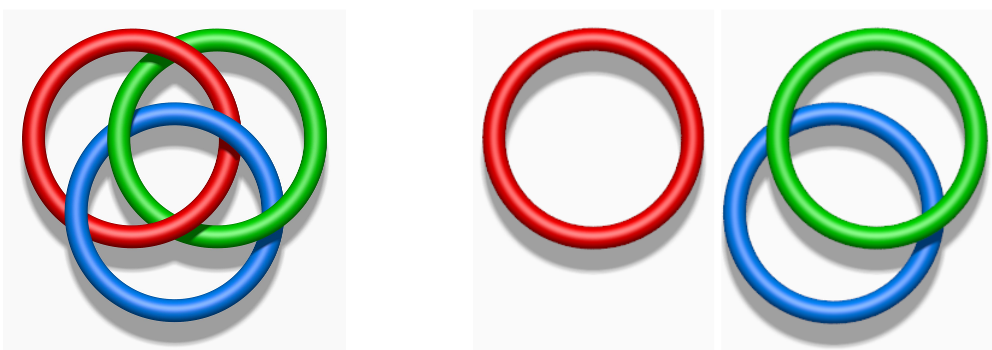

**Figure 1** : *Borromean rings, optical illusion [1] (left); one ring removed (right)*

The Borromean rings structure possesses, of itself, already an intriguing and artistic character. Moreover, it has, like many other mathematical structures such as regular tilings, lattices, symmetry groups, polyhedra, and knots, also inspired artists for many years to create all kinds of variations. In this paper, we present three families of sculptures inspired by the Borromean rings. A common characteristic of these sculptures is that all have been constructed from *straight* beam segments connected by *miter joints*. These constraints pose their own challenges [10]. In the first family (§2), the loops are rendered as simple hexagonal rings. They stay closest to the original. The second family (§3) arose from weaving a fabric of beams around a cube. On each face, two sets of beams weave over-and-under, connecting to the beams on the next face, forming three sets of jagged loops. Finally, the third family (§4) appears when braiding three beams along a triangular path, and closing it such that three separate, but intertwined, loops are obtained. Section 5 concludes the paper, including an overview of related work by others.

## 2 First Family, Based on Convex Planar Hexagons

It is well-known [7] that the Borromean rings cannot be constructed using three circles embedded in space. However, three ellipses will work, and in fact they can be pairwise orthogonal in a highly symmetric arrangement (see Figure 2, left). When constructing rings from a number of straight beams connected by miter joints, we can only approximate the nature of the ellipse. Obvious, albeit dull, candidates then are a rectangle or rhombus, since it is longer than wide (see Fig. 2, middle). It can also be accomplished with equilateral triangles (and even with squares), but that configuration is less symmetric (see Fig. 2, right).

**Figure 2** : *Borromean ellipses (left), Borromean rectangles in wood (middle), Borromean triangles (right)*

A more interesting shape that is naturally longer than wide is the regular hexagon (see Fig. 3, left). How thick can the beams of the hexagons be? Neighboring rings may touch but not intersect. Thereby, all internal degrees of freedom will be removed. Figure 3 (right) depicts part of one regular hexagon and the two cross sections of the enclosed hexagon (the two circles). Assuming that the center lines of the beams have length 1, we have $OA = 1$ and $OC = OD = \frac{1}{2}\sqrt{3}$. Hence, $CA = OA - OC = 1 - \frac{1}{2}\sqrt{3} \approx 0.134$ and moreover $CB = 2r$, where $r$ is the beam radius. Since $\angle ACB = 30°$, we find $r = -\frac{3}{8} + \frac{1}{4}\sqrt{3} \approx 0.058$.

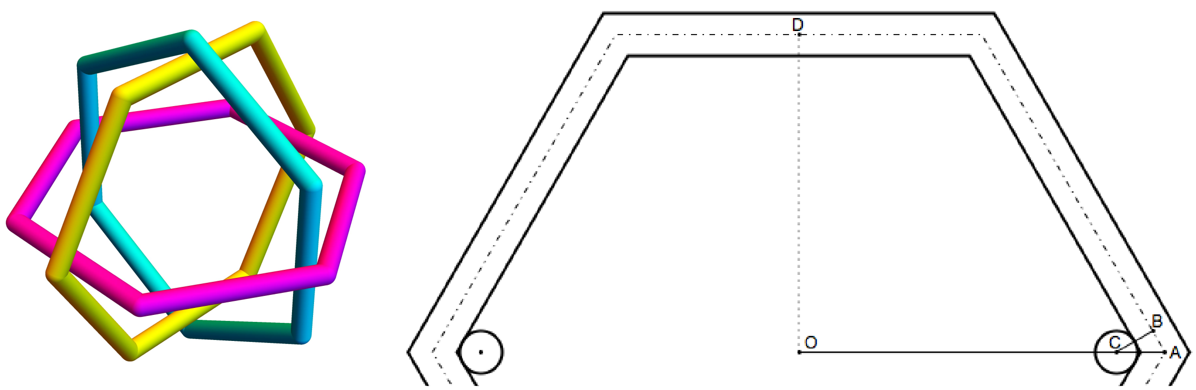


**Figure 3** : *Borromean regular hexagons with round cross section (left), cut through (right)*

Note that there is a little bit of empty space where one beam joint bends around another beam (see Fig. 3, between $C$ and $A$). The next idea is to use beams with a cross section such that the beam faces leave no empty space between the beams. This requires a triangular cross section (see Fig. 4). The maximized cross section turns out to be an isosceles triangle with a top angle of $120°$ (see the two light triangles labeled $C$ just inside the fully shown hexagon on the left in Fig. 4). A similar calculation as with hexagons from round beams shows that the cross section's base is $3 - \frac{3}{2}\sqrt{3} \approx 0.4$, when the outer length of the beams is 1. The cut faces are isosceles triangles with the same base, but a top angle of $2\arctan(3/2) \approx 112.6°$ (see the two darker triangles labeled $J$ just outside the fully shown hexagon on the left in Fig. 4).

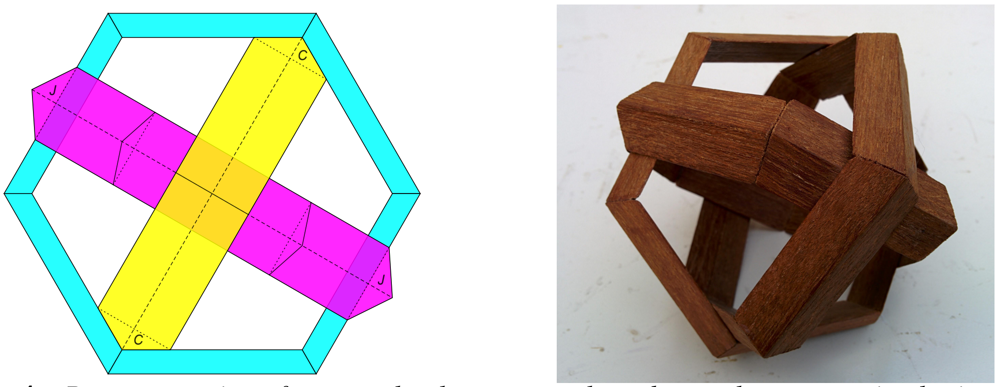


**Figure 4**: *Borromean rings from regular hexagons, whose beams have a maximal triangular cross section: projection (left), wooden sculpture (right)*

In order to tighten the design and reduce the eight 'holes', we can give up the requirement that all beams are equally long, still keeping all angles equal, i.e., $120°$. The cross section and the cut faces keep the same shape; only the beams are shortened (see Fig. 5).

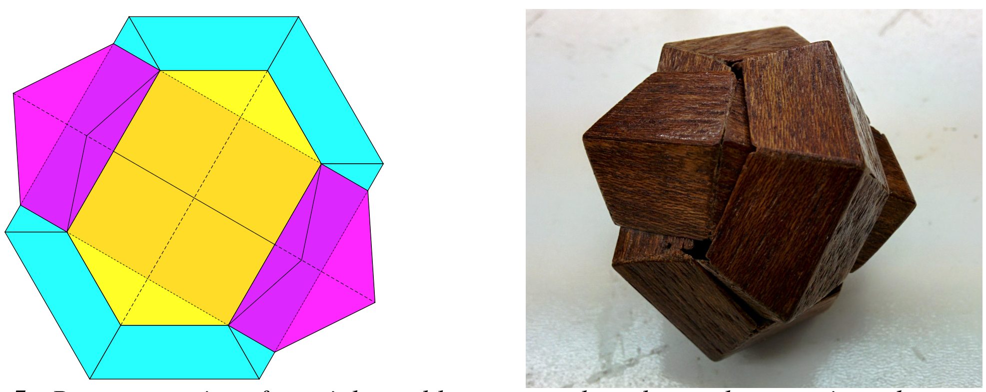

**Figure 5**: *Borromean rings from tightened hexagons, whose beams have a triangular cross section: projection (left), wooden sculpture (right)*

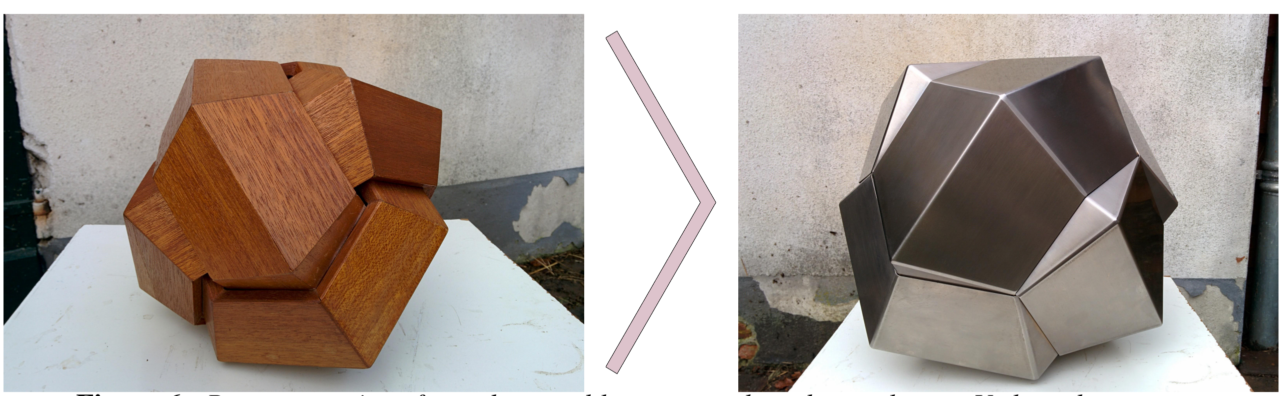

**Figure 6**: *Borromean rings from elongated hexagons, whose beams have a V-shaped cross section: wooden sculpture (left), beam cross section (middle), stainless steel sculpture (right)*

When constructing a larger version, and especially when doing so in stainless steel, the triangular cross section is inconvenient. This can be solved by using a V-shaped cross section, consisting of the two equal sides of the triangular cross section in the preceding designs (see Fig. 6 and Fig. 16, left). This cross section can be obtained by gluing (wood) or welding (steel) two strips side by side at an angle of $120°$.

Other variants arise when also dropping the requirement that all angles of the hexagons are equal, but keeping opposite sides parallel and the symmetry group of the rectangle. For instance, if you have a beam with an isosceles right triangle as cross section, then hexagons with angles of $90°$ $(2\times)$ and $135°$ $(4\times)$, and two different side lengths will work (see Fig. 7a). This design has a 'tip-tilted nose': the cut face of the overpass is not a right triangle, implying that its outer edge is almost but not exactly parallel to the near edge of the hexagon that it passes over (the two edges along the top the image). This can be corrected by using as cross section an isosceles triangle with top angle of $98.03°$ (see Fig. 7b). Another obvious candidate for the cross section is an equilateral triangle (see Fig. 7c). Because of the regular miter joints, the underpassing beams of the hexagons stick out slightly with respect to the overpassing beams ('shoulders': see inset). This can be corrected by using skew miter joints [10], where the underpass is cut off at a right angle (so, not along the internal bisector plane of the angle). Consequently, its cross section no longer can be an equilateral triangle (see Fig. 7d). Thus, all gaps disappear and the 24 exposed faces nicely share the concave edges. In fact, the 30 invisible faces can be omitted.

(a) (b) (c) (d)

**Figure 7**: *Wooden Borromean ring sculptures from elongated hexagons, whose beams have as cross section: (a) an isosceles right triangle, (b) isosceles triangle with top angle of* $98.03°$*, (c) equilateral triangle (inset shows one 'shoulder'; cf. Fig. 16, middle), (d) equilateral triangle and skew miter joints*

The final design in this hexagon-based family was driven by the desire to make the faces around each hole co-planar, resulting in an octahedral shape (see Fig. 8). Two angles of the hexagon are $90°$; hence, two of the beams have a right triangle as cross section (which passes underneath the right angle of the hexagon). The other four beams have an isosceles triangle with top angle of $2\arctan\sqrt{2} \approx 109.5°$, the dihedral angle of the octahedron. Thus the segments differ in cross section, while neatly being connected by skew miter joints. It is called Trinity II (for Trinity I, see [3]).

**Figure 8** : *Trinity II: Borromean rings from elongated hexagons, using skew miter joints, having an overall octahedral shape (left), one link (right)*

## 3 Second Family, Based on Weaving around a Cube

The second family actually did not arise directly from an attempt to render the Borromean rings. It started with the observation [12, Fig. 2] that two orthogonal sets of right-angled zigzag beams with a $1 : \sqrt{2}$-rhombic cross section can be woven into a tight fabric (see Fig. 9). Note that the cut faces at the joints are squares that snugly fit the crossing $90°$-joints. The strands of this weaving can be closed by wrapping them around a

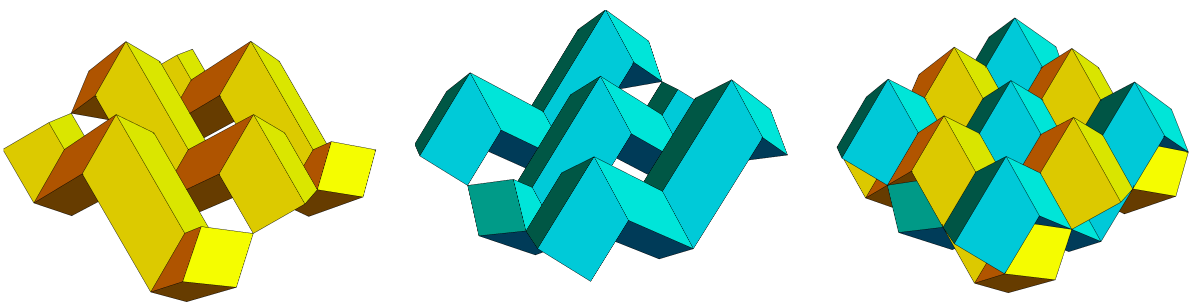

**Figure 9** : *Weaving two orthogonal sets (left, middle) of three* $90°$ $\sqrt{2}$*-rhombic zigzags into a tight fabric (right)*

cube (see Fig. 10). To cover the entire cube you need three sets of parallel strands. This results in nine loops, no pair of which is linked. Yet, together they form a generalized Borromean link, known as Brunnian link. In general, an odd number of parallel strands per set leads to a Brunnian link. If an even number of strands is used per direction, then orthogonal pairs are linked, and the structure is not Brunnian (see Fig. 11).

**Figure 10** : *Brunnian weaving around the cube, with three bands per direction (three different views)*

**Figure 11** : *Weaving around the cube, with four bands per direction, non-Brunnian (three different views)*

A degenerate case of this weaving around the cube uses only one strand per direction. The result is a Borromean link, called Trinity III (see Fig. 12, left). Its overall shape is that of an octahedron, bearing a strong resemblance to Trinity I and Trinity II. Each ring is now a planar, non-convex octagon, (see Fig. 12, right), whose double square hole is exactly filled by two square cut faces of another link. Note that each pair of octagons is topologically not linked, but when they are rigid, they still cannot be separated.

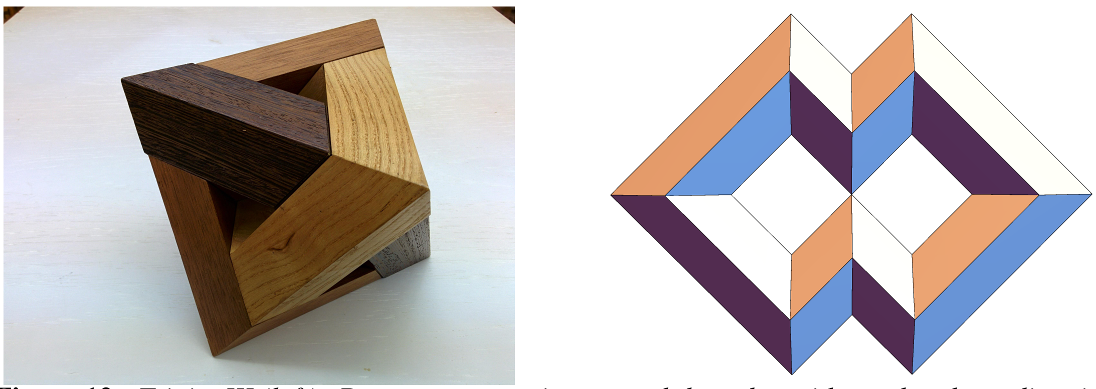

**Figure 12** : *Trinity III (left): Degenerate weaving around the cube, with one band per direction, each band being a non-convex planar octagon (right)*

## 4 Third Family, Based on Braiding

The well-known three-stranded hair braid already has a Brunnian character: each strand lies strictly above another strand and below the third. Such a braid can be mimicked by straight beam segments joined at right angles, as can be recognized in the closed triangular braids of Fig. 13. These closed braids were designed

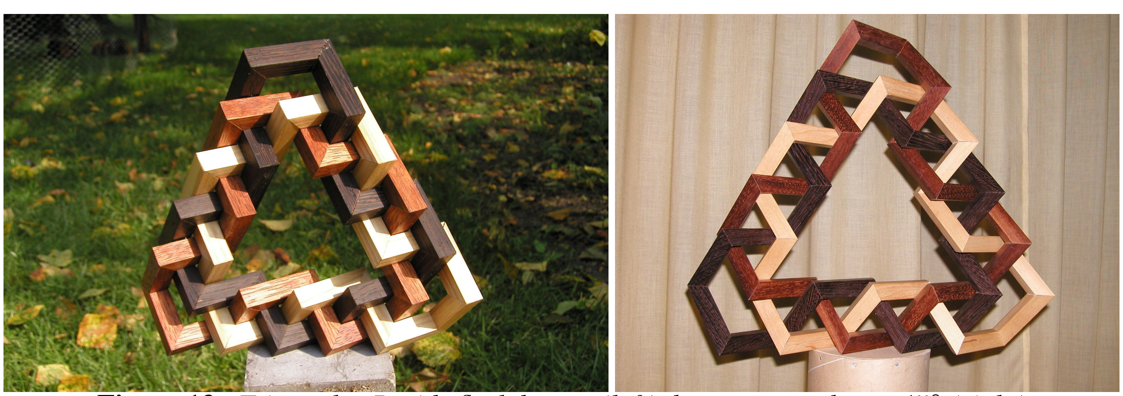

**Figure 13** : *Triangular Braid: flush beams (left), beams rotated over* $45^\circ$ *(right)*

such that three separate loops arise that are identical in shape. This shape is described in detail in [11] using 3D Turtle Geometry. The triangular braid can be extended indefinitely since it is periodic. Shortening it yields a degenerate triangular braid (Fig. 14, left), which turns out to consist of three nonplanar hexagons forming a Borromean link. The nonplanar hexagons can be varied in shape to yield the Borromean sculpture on the right in Fig. 14.

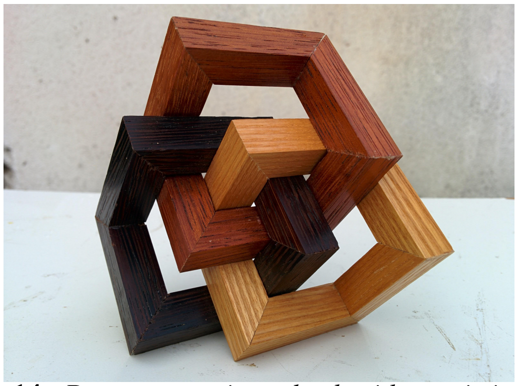
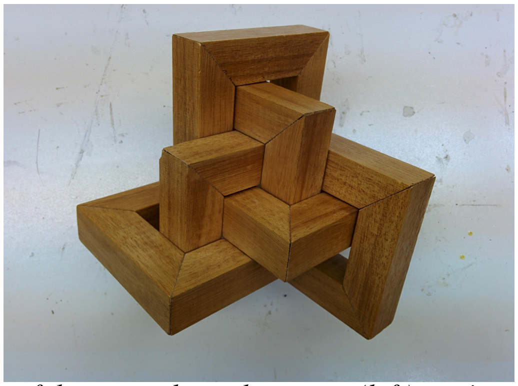

**Figure 14** : *Degenerate triangular braid, consisting of three nonplanar hexagons (left), variant (right)*

## 5 Conclusion

We have explained the design of three families of Borromean ring sculptures using straight beam segments connected by miter joints. The underlying ring shapes are planar hexagons, planar octagons, and nonplanar hexagons. The odd-weaving and the braiding technique, in general, yield Brunnian structures. Also worth mentioning are two plane-spanning Borromean structures. The sculpture on the left in Fig. 15 is based on 12-gons that link in Borromean style. On the right is a weaving of zigzags from rhombic beams that cross in three directions. If one set of parallel zigzags is removed, the other two can easily be separated by lifting.

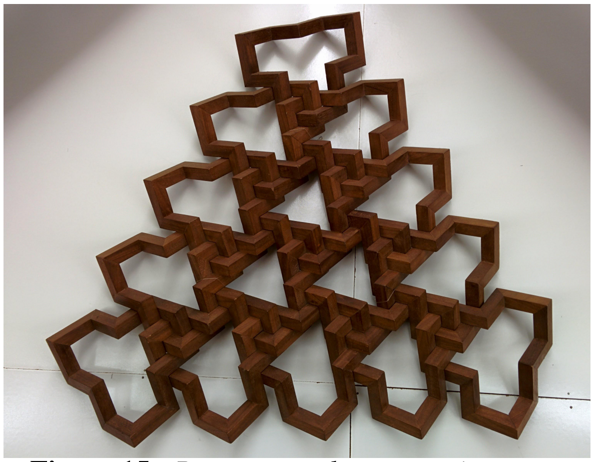


**Figure 15** : *Borromean plane-spanning structures: nonplanar 12-gons, also see right in Fig. 16 (left); zigzags in three directions (right)*

**Related work** Holden presents various Borromean designs in [5]. The Borromean theme can be recognized in many artworks, including some traditional African wood carvings [2]. We particularly like John Robinson's *Creation* (Borromean squares), *Intuition* (Borromean triangles), and *Genesis* (Borromean rhombi) [8]. Jablan investigates the rarity of Borromean designs in [6]. Séquin describes a way of splitting a torus into Borromean triplets [9]. The logo of the International Mathematical Union (IMU) is Borromean [4].

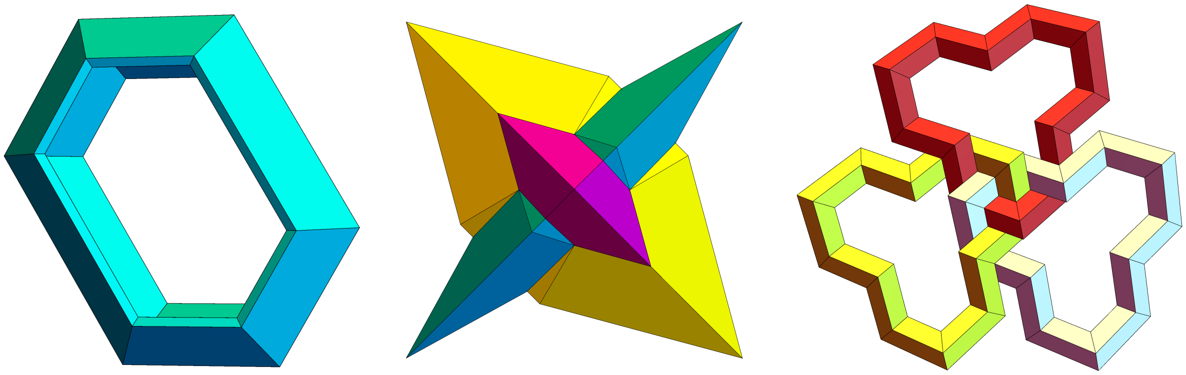

**Figure 16** : *Hexagon with V-shaped beam, cf. Fig. 6 (left); better view of the 'shoulders' in Fig. 7c (middle); Borromean link with nonplanar 12-gons, cf. left in Fig. 15 (right)*

**Future work** We are still looking for ways to close the three-directional weaving of Fig. 15 (right) by wrapping it appropriately around some polyhedron.

**Acknowledgments** All sculptures in the figures were designed by Koos Verhoeff. The wooden sculptures he constructed himself. The stainless steel sculpture in Fig. 7 was constructed by Geton, Veldhoven, The Netherlands. The design calculations and diagrams were done with Mathematica.

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