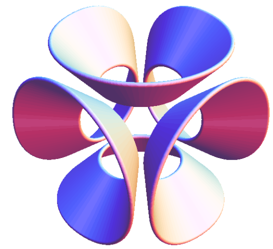
Lobke, and Other Constructions from Conical Segments

Tom Verhoeff *
Department of Mathematics and Computer Science
Eindhoven University of Technology
Den Dolech 2
5612 AZ Eindhoven, Netherlands
`T.Verhoeff@tue.nl`

Koos Verhoeff
Valkenswaard, Netherlands



Abstract

Lobke is a mathematical sculpture designed and constructed by Koos Verhoeff, using conical segments. We analyze its construction and describe a generalization, similar in overall structure but with a varying number of lobes. Next, we investigate a further generalization, where conical segments are connected in different ways to construct a closed strip. We extend 3D turtle geometry with a command to generate strips of connected conical segments, and present a number of interesting shapes based on congruent conical segments. Finally, we show how this relates to the skew miter joints and regular constant-torsion 3D polygons that we studied earlier.

1 Introduction

Figure 1 : *Lobke, a mathematical sculpture consisting of six conical segments*

Figure 1 shows *Lobke*, a mathematical sculpture designed and constructed by Koos Verhoeff in the early 1990s. It is made from fiberglass with polyester resin on a metal mesh, and currently located in the sculptor's garden. The name is an affective diminutive of 'lob', the Dutch word for 'lobe'. *Lobke* can be decomposed into six congruent conical segments connected into one smooth continuous two-sided strip. However, there is a practical problem to address. To understand this, we analyze the construction in more detail.

The basic idea is simple. Place six *cones* inside a *cube*, their apexes meeting in the center, and their base circles being the incircles of the cube's faces (Figure 2, left). The cone's aperture (angle at the apex) is 90°. A path on the surface of these cones can switch from one cone to another at the lines where two cones touch each other. By switching cone after three-quarters of a cone, one can trace a closed path that travels on all six cones. Instead of a path, we take six three-quarter conical segments (Figure 2, right). The practical problem is that this closed (infinitely thin) strip self-touches in six places (at one-third and two-third). When thickening the mathematical strip to construct a physical object, it will self-intersect.

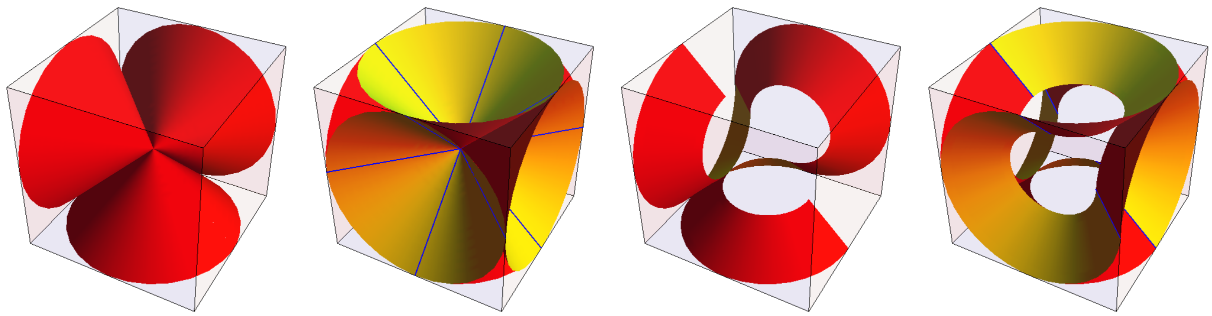


Figure 2: *Touching cones in a cube (left: three and six cones with tangent lines); connected three-quarter conical segments in a cube (right: three and six segments with cuts)*

To avoid self-intersection, the parameters of the conical segments must be adjusted. However, we want a strip where the segments are connected smoothly, that is, two connected conical segments share a tangent at the joint. Because of *Lobke*'s sixfold symmetry ($60°$-*rotoreflection*; see Figure 3), the six cuts (where conical segments meet) must make $60°$ angles, and when extended they intersect in the center. So, we cannot just shrink the cones towards the cuts. The only thing we can vary of the cone is its aperture (Figure 4, left; compare to Figure 6 of [6]; the base circles of all these conical segments lie neatly on a sphere). Reducing the aperture from $90°$ to $86°$ provides sufficient clearance between the lobes to thicken them (Figure 4, middle). The smallest aperture that still works is, obviously, $60°$ (Figure 4, right). The base circles of the lobes are no longer incircles of the faces of a cube, but of a *trigonal trapezohedron*, whose faces are six congruent rhombuses. In case of a $60°$-aperture, it degenerates into a *uniform hexagonal prism*.

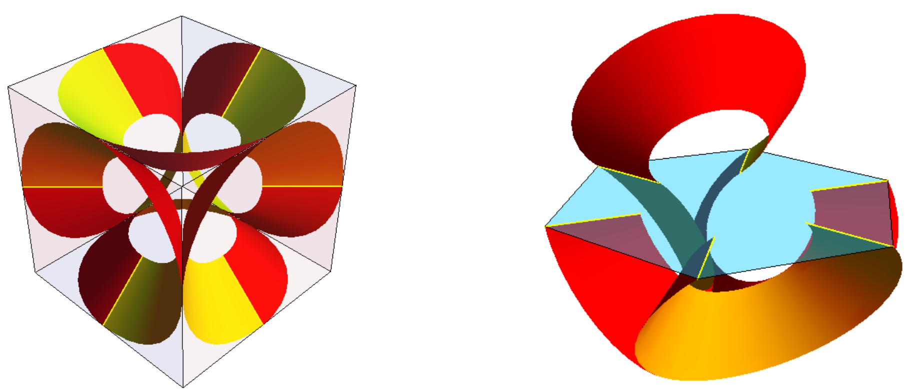


Figure 3: *Sixfold rotoreflective symmetry: the cuts make* $60°$ *angles, intersecting in the center.*

2 First Generalization of Lobke: Varying the Number of Lobes

A straightforward generalization of *Lobke* is to vary the number of lobes. With three lobes above and three lobes below, there are six cuts, making angles of $60°$. In general, we can have k lobes above connected to another k lobes below, giving rise to $2k$ cuts, making angles of $180°/k$. The aperture and cone fraction have to be chosen appropriately. Figure 5 shows variants with four, eight, and ten lobes. The base circles are the incircles of a k-*gonal trapezohedron*. Self-intersecting variants are also appealing (Figure 6).

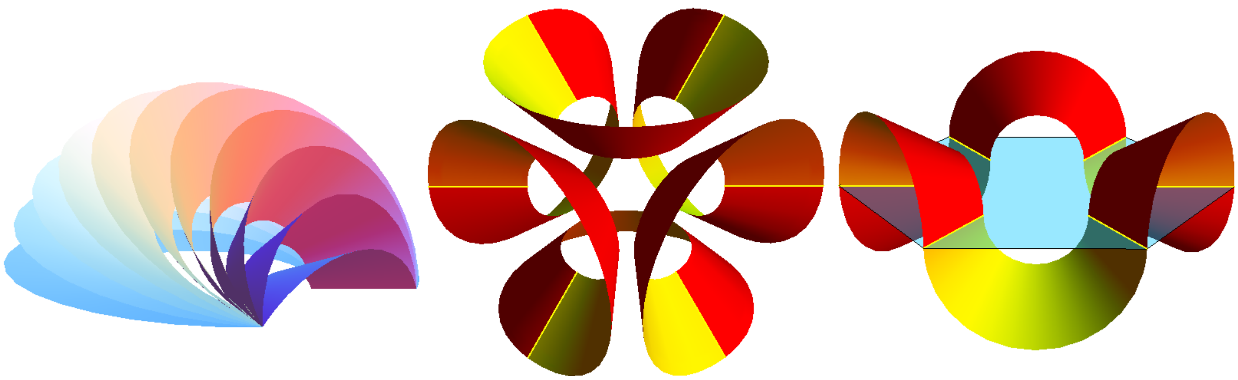


Figure 4 : *Conical segments sharing cuts at* $60°$ *and varying aperture (left),* $86°$ *(middle), and* $60°$ *(right)*

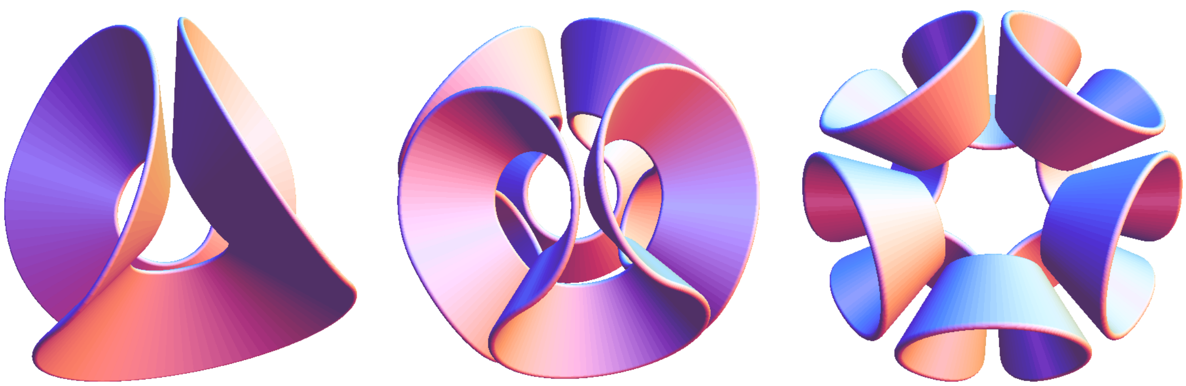


Figure 5 : *Three Lobke variations, with four, eight, and ten non-touching lobes*

3 Second Generalization of Lobke: Varying the Connections

Another generalization is obtained by connecting the conical segments in different ways. Note that two conical segments can be connected smoothly in four ways (Figure 7). In *Lobke* and its first generalization, all connections are of type 2, where the edge circles of the same radius meet each other at the joint, and where the inside surface (toward the axis of the cone) of one segment meets with the outside surface of the other (and vice versa). Type 1 is an inversion (dual) of type 2, where the meeting edge circles differ in radius,

Figure 6 : *Three Lobke variations, with four (reminiscent of the Enneper surface), six, and eight self-intersecting lobes*

and where the inside surfaces meet. Type 3 just extends the conical segment, that is, edge circles with the same radius meet, and inside meets inside. Finally, type 4 is the opposite of type 3; here, the edge circles that meet differ in radius, and the inside surface of one segment meets the outside surface of the other.

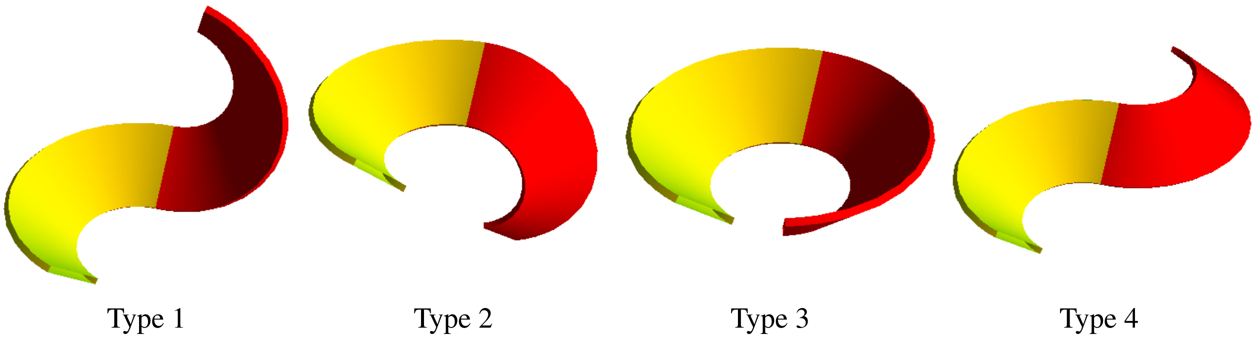


Figure 7 : *The four ways to connect two conical segments smoothly*

The question now arises as to which configurations of smoothly connected conical segments close properly. Also self-intersection can be a concern. Ad hoc trial and error yielded some initial discoveries (Figure 8), but ensuring that a design closes properly is not so easy.

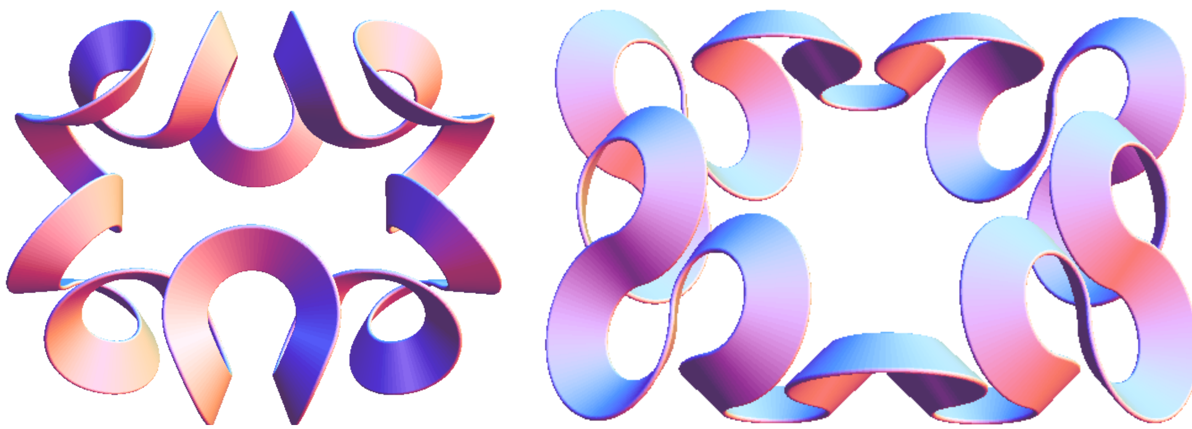


Figure 8 : *Two ad hoc closed designs by smoothly connecting conical segments in multiple ways*

Because there are so many possibilities, a more systematic approach is desirable. To describe these strips of connected conical segments in a convenient way, we extended the 3D Turtle Geometry of [5]. As the turtle moves, it generates a strip, whose direction and normal vector equal the turtle's direction and normal vector. A new command $\mathit{CStrip}(\alpha, r, \beta)$ makes the turtle move on a cone whose axis makes an angle α with the turtle's ground plane (aperture 2α), having base radius r, turning over angle β along the cone's base circumference ($\beta = 360°$ is a full turn). Figure 9 shows the results of three *CStrip* commands. The turtle's initial and final state are labeled O and F. The heading, port-side (left-hand), and normal vector are labeled h, p, and n. The degenerate case $\alpha = 90°$ results in a turn on a disk (a cone with aperture 180°). Taking $\alpha = 0$ is another degenerate case, resulting in a turn on a cylinder (a cone with aperture 0).

A strip of smoothly connected conical segments can then be described by a sequence of *CStrip* commands. For instance, *Lobke* is described by the following six commands (the superscript denotes repetition):

$$\left(\mathit{CStrip}(\alpha, r, \beta)\,;\,\mathit{CStrip}(180° - \alpha, r, \beta)\right)^3 \tag{1}$$

where $\alpha = 43°$, $r = 1$, and $\beta = 265.7°$. With $0 < \alpha < 90°$, the strip from command $\mathit{CStrip}(\alpha, r, \beta)$ curls *up* to the *left* (cf. Figure 9, left). By taking α beyond that range, the curl direction will vary: flat-left

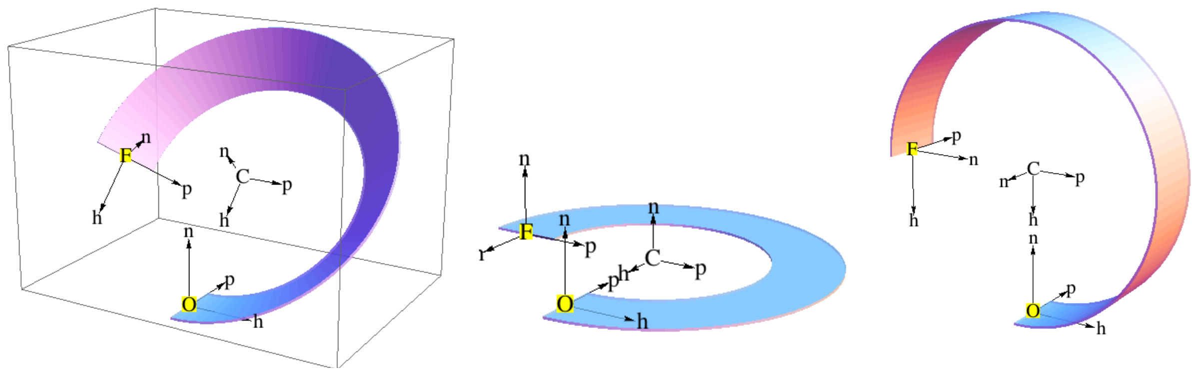


Figure 9 : *Turtle that executed* $\mathit{CStrip}(45°, 1, 270°)$*,* $\mathit{CStrip}(90°, 1, 270°)$*,* $\mathit{CStrip}(0, 1, 270°)$

($\alpha = 90°$), down-left ($90° < \alpha < 180°$), down-straight ($\alpha = 180°$), down-right ($180° < \alpha < 270°$), flat-right ($\alpha = 270°$), up-right ($270° < \alpha < 360°$), up-straight ($\alpha = 0$). Thus, for $0 \leq \alpha \leq 90°$, the following four commands (preceded by an index, for later reference) concern congruent cones with aperture 2α:

0. $\mathit{CStrip}(\alpha, r, \beta)$,
1. $\mathit{CStrip}(180° - \alpha, r, \beta)$,
2. $\mathit{CStrip}(180° + \alpha, r, \beta)$,
3. $\mathit{CStrip}(360° - \alpha, r, \beta)$,

Hence, these four conical strip segments are congruent. The shape of a strip constructed just from congruent conical segments with fixed aperture, radius, and turn angle is then fully described by α, β, and the sequence of *indices* of the commands in the numbered list above (0 through 3). From Formula (1) we see that in *Lobke*, the strip alternates up-left and down-left segments. Thus, the index sequence for *Lobke* is

$$(0, 1)^3 \tag{2}$$

For the strips in Figure 8, the index sequences are respectively

$$(0, 1, 2, 3, 2, 1)^3, \text{ with } \alpha = 46.27°, \beta = 225° \tag{3}$$

$$(0, 2, 1, 3, 1, 2)^3, \text{ with } \alpha = 45°, \beta = 270° \tag{4}$$

In the latter strip, the turtle's *waypoints* between *CStrip* commands, i.e., where adjacent segments join, lie in a lattice, making it easier to ensure proper closure. Another way to find properly closed strips is by experimentation: fix an index sequence, and vary α and/or β until proper closure occurs. Figure 10 shows the index sequence of (3) with $\alpha = 36°$ and β stepping from 245° to 247°. For a given index sequence, proper closure is not always possible. This procedure resembles our approach to finding *regular constant-torsion polygons* in 3D, as described in [4]. We do not have a general theory to achieve proper closure.

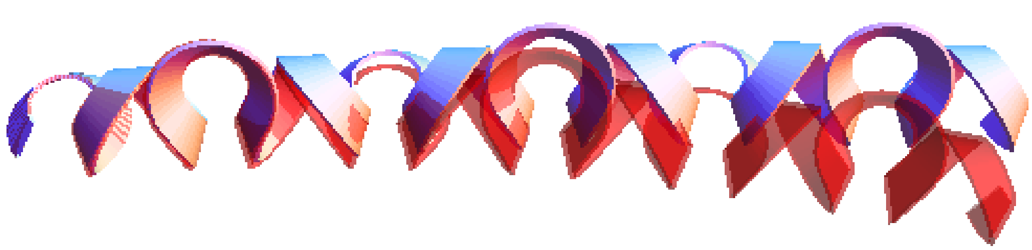


Figure 10 : *The strip generated by the sequence* $(0, 1, 2, 3, 2, 1)^3$ *with* $\alpha = 36°$ *and* $\beta = 246 \pm 1°$*, cf. (3)*

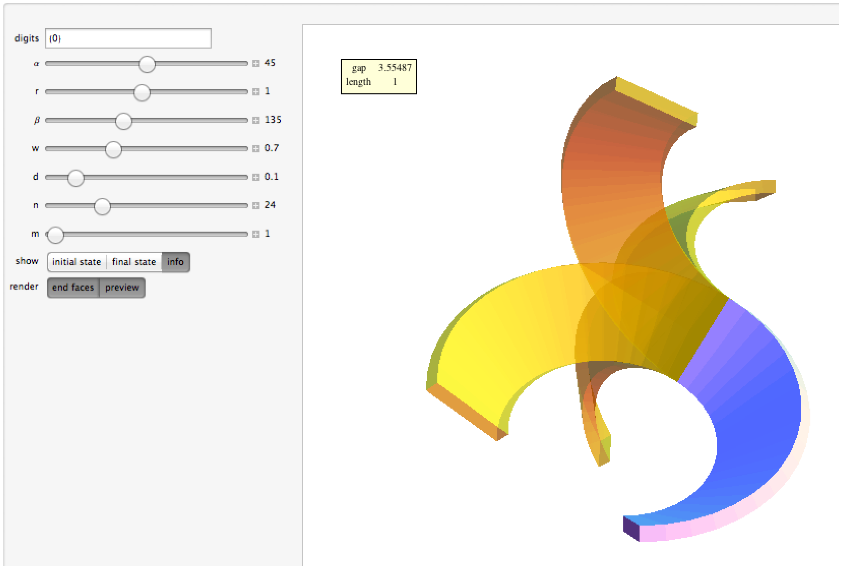


Figure 11 : *GUI for conical segment strip explorer in Mathematica*

We developed an interactive explorer of conical segment constructions in Mathematica (Figure 11). You can set various parameters, such as α, r, β, and the strip's width and thickness (these are the same for all segments). A sequence of segments is interactively built up, where you select one of the four possible continuations shown half transparent. As an aid to achieving proper closure, the distance between the initial and final state is shown, taking difference in attitude into account.

4 Relationship with Skew Miter Joints

Further exploration resulted in numerous interesting shapes (Figure 12). At some point, it dawned upon us that certain shapes were somehow familiar. Note that, in the images, the cones are approximated by a large number of small straight steps. When reducing the number of steps in the approximation, a metamorphosis takes place. Figure 13 shows this metamorphosis for the shape on the left in Figure 12. If the strip in the straight-line approximation is thickened, so that thickness and width become equal, then the shape on the left in Figure 14 is obtained. Similarly, the other shapes in Figure 12 correspond to shapes in Figure 14. The one in the middle, a Hamilton cycle on the cuboctahedron, appears in a different orientation.

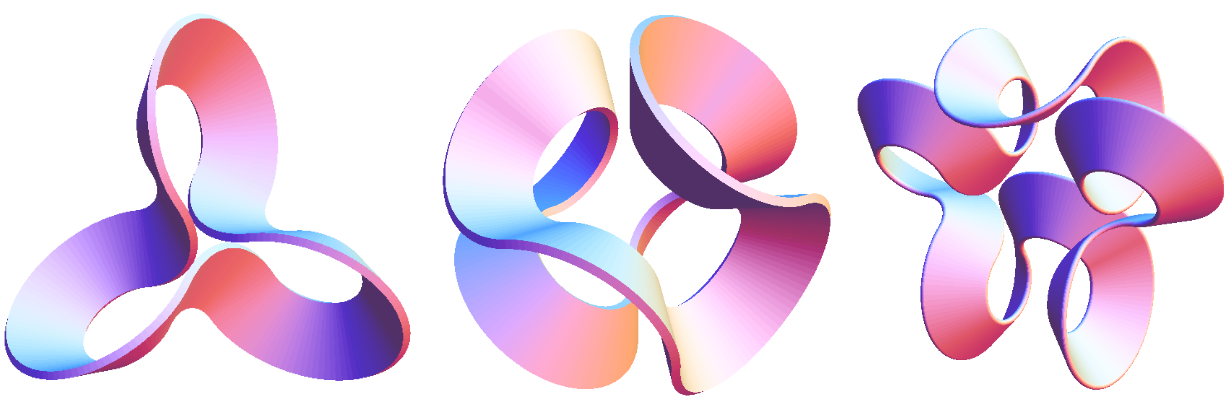


Figure 12 : *Three properly closed constructions from conical segments (left: note the Möbius twist)*

Figure 13 : *The same shape, where the conical segments are approximated in fewer (3, 2, 1) steps*

Figure 14 : *Conical segments in Figure 12 approximated by straight trapezoidal beam segments*

The shapes in Figure 14 are well-known to the authors. The shape on the left appears as one of the two links in *Hopeless Love I* [7, Picture 14]. The two on the right appear in [3], in Figures 9 and 10 to illustrate artwork involving *skew miter joints*, where the cut face does not lie in the angle bisector plane (also [7, Pictures 19 and 11]). Thus, all designs involving trapezoidal beam segments can be converted into constructions with conical segments. Some further designs obtained in this way are shown in Figure 15.

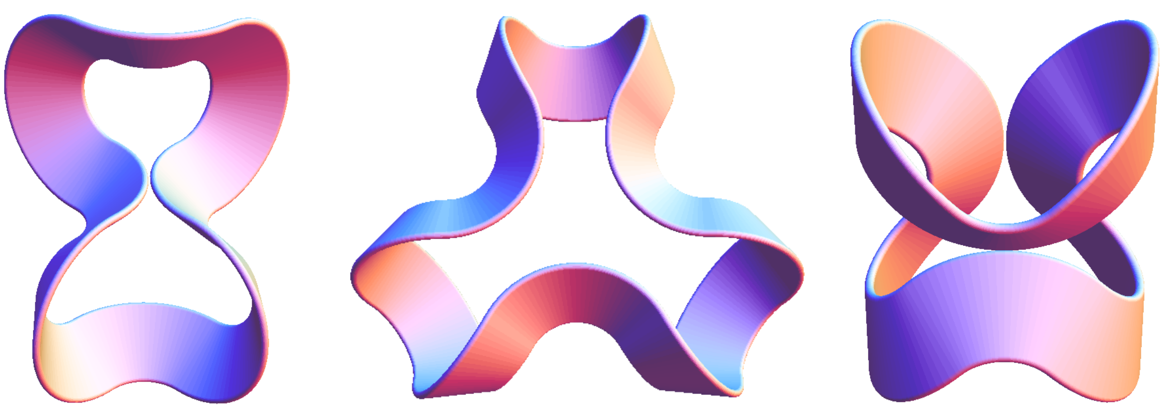


Figure 15 : *Three more properly closed constructions from conical segments*

5 Conclusion

We have explored several variations on the mathematical sculpture *Lobke* constructed from conical segments, resulting in interesting new shapes and questions. We introduced a notation based on 3D Turtle Geometry to describe shapes constructed from smoothly connected conical segments. We do not know under what conditions such strips are properly closed. These conditions seem to resemble the conditions for regular constant-torsion polygons in 3D. There also turned out to be a surprising relationship with skew miter joints.

Related work Several other artists have created sculptures involving (near-)conical segments: *Seat of Wisdom* and *Circle Squared* by Vic Pickett; *Bronze Spheric Theme* and *Model for 'Spheric Theme'* by Naum Gabo; *Snake*, *Berlin Junction*, and other sculptures by Richard Serra (see [1]). However, we have found no mathematical analyses of these sculptures. *Borsalino* and other sculptures by Henk van Putten use cylindrical segments with a square cross section (see [2]). *Arabesque XXIX* by Robert Langhurst resembles *Lobke*, but it has no hole and it is not a developable surface.

Future work A strip of conical segments can be viewed as a rotation-minimizing sweep along a 3D space curve composed of circular segments. This may provide another handle on proper closure. It is possible to rotate the strip along its center line. In general, if all conical segments are congruent, then after such a rotation, they need no longer be congruent (see Figure 12, left, and Figure 16, left), but they do remain conical segments. In the special case of a strip that is as wide as it is thick (that is, with a square cross section), there are further ways to connect them smoothly that we have not investigated (see Figure 16, right, and [2]). Since a cone is a developable surface, a strip constructed from conical segments can be flattened onto the plane. What strips can be flattened without self-intersection?

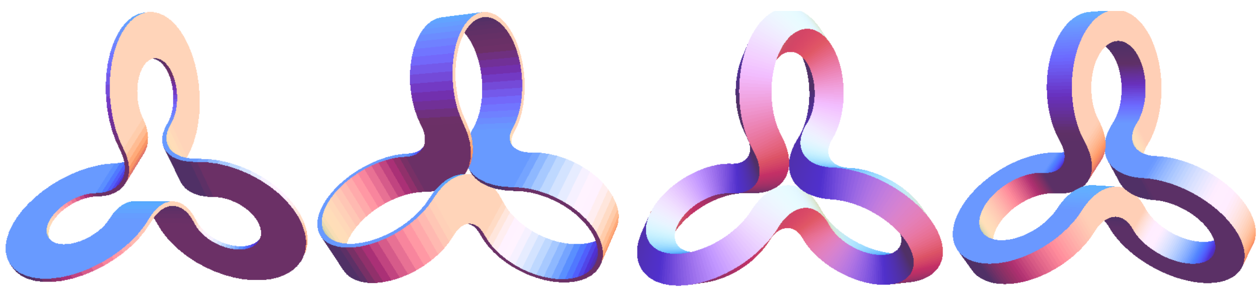


Figure 16: *Conical segments rotated about the center line (left two); square cross section (right two)*

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