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"WHICH SUBJECTS IN MODERN MATHEMATICS AND WHICH APPLICATIONS IN MODERN MATHEMATICS CAN FIND A PLACE IN PROGRAMS OF SECONDARY SCHOOL INSTRUCTION?" [1]

by

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1. *Preface*

The International Commission on Mathematical Instruction chose the present topic as one to be studied by national subcommissions in the years 1958 to 1962. When I learned that I was to serve as reporter for this topic at the Stockholm Congress, I contacted all the national subcommissions of ICMI, requesting that reports be sent to me when available. I am very pleased to note that I am now in possession of 21 national reports from all over the world. The following is a summary of these 21 reports, with special emphasis on similarities and differences in points of view.

While I am taking every possible precaution to represent views of various nations accurately and fairly, I fully realize that brief reports cannot reproduce accurately many long years of work. May I therefore take this opportunity to apologize to any mathematician who may feel that the following report is either inaccurate or an insufficient presentation of achievement in his own nation.

2. *The process of change*

Only a very few countries reported that so far little or no attempt to introduce modern mathematics had taken place. Of course, this small number may not be significant, since my sample is biased: Presumably countries in which absolutely no attempt to modernize mathematics has occurred have not filed reports on this topic.

Of the remaining countries, the vast majority report that the attempts to modernize the curricula have consisted mostly of informal discussions amongst mathematics teachers and a number of highly encouraging experiments by individual teachers. It seems

[1]) Report delivered at the International Congress of Mathematicians, Stockholm, 1962.

to be a universal experience that attempts to teach selected topics from modern mathematics well, in reasonable quantities, can be highly successful.

I shall discuss in somewhat more detail reports of a few countries where national reform movements have taken place.

France had a head-start over most other countries in that the French secondary school mathematics program was even traditionally unusually strong. The typical secondary school teacher in France had a strong university degree in mathematics which both placed France into a good starting position and made it easier to introduce modern ideas. Reform started with a series of experiments by teachers trying out various topics of modern mathematics in the classroom. This led to the writing of a series of articles and monographs which were widely discussed. Eventually, a number of seminars were formed at which secondary school teachers and college professors together discussed pedagogical problems involved in curriculum reform. France is fortunate enough to have persuaded a number of its very famous mathematicians to give lectures to high school teachers on topics of modern mathematics. This is all the more remarkable, in that all of this work, both on the part of the lecturers and the high school participants, was entirely voluntary without compensation. All of this effort finally resulted in success: The Ministry of Education gave its official blessing to plans formulated for the modernization of the secondary school curriculum.

There is also a report of some experiments in France with children of a younger age, to present some basic ideas of geometry, number, and sets from a modern point of view.

Curriculum reform in Germany is complicated by two factors. First of all, in the Federal Republic the problem of education is not in the hands of the Federal Government but of the individual States. Therefore, it is very difficult to initiate a national reform. A permanent conference of ministers of education has been established to provide some degree of uniformity in school curricula. A second complicating factor is the existence of three types of gymnasiums in Germany, with quite different attitudes towards the teaching of mathematics. Real reform has been possible primarily in the mathematics-science version of the gymnasium.

On the other hand, the German gymnasium covers a nine-year period and therefore can provide a continuity in mathematical instruction not possible in most other countries. The German report points out a problem common to many nations — that the amount of time allocated to mathematics in the curriculum is severely limited. Therefore, the introduction of modern mathematics cannot be thought of as the addition of new topics to an existing curriculum. Rather, one must find topics within the traditional curriculum which, although they may have been worthwhile, are not from a modern point of view indispensable. Modern ideas are introduced by the replacement of such topics with selected ideas from modern mathematics. On the other hand, one often has an opportunity to supplement these topics for the better students in an "Arbeitsgemeinschaft", where students voluntarily go deeper into the subject matter. Apparently such informal courses play an important role in the education of mathematics students in Germany. Not only does Germany propose a new curriculum for high school mathematics, but their report shows evidence of deep thinking on individual topics in this curriculum. A number of extremely useful articles and monographs have been written in Germany, and the reader will find in the appendix of this report a bibliography from the German report.

The status of Italy seems typical of a large number of countries. Two national commissions have studied the problem of modernizing the high school curriculum, and have reported their findings. Italy is now ready to start implementing these recommendations.

In Israel the Ministry of Education has appropriated funds for the writing of experimental textbooks by a group of mathematicians at Hebrew University.

Poland is an example where, although relatively little actual experimentation has been done in the classroom, there has apparently been an immense amount of highly constructive discussion amongst the teachers of mathematics. The Polish report gives every evidence of having had topics discussed both in a wide range and in great depth; and of highly laudable, constructive thought on the part of many mathematicians. The report indicates that these plans have now reached the stage where they hope to try out experiments on a variety of different lines in the classroom.

A most interesting cooperative enterprise is under way in the Scandinavian countries. They have formed a "Scandinavian Committee for the Modernizing of School Mathematics". This represents a cooperative effort amongst Denmark, Finland, Norway and Sweden to pool their resources, both mathematical and financial, for the improvement of mathematical education. This is made possible not only by the geographic proximity of these countries but by strong similarities amongst their educational systems, as well as traditional ties.

In 1960 the Committee adopted a 5-point program: (1) To survey mathematical needs both for the use of industries and for the needs of universities. (2) The development of new mathematical curricula. (3) The writing of experimental textbooks. So far four monographs have been produced. (4) Plans have been made for extensive testing of these experimental materials. (5) After these tests have been concluded, the Committee is to make official recommendations to the four governments for the adoption of new curricula for secondary education.

The United States has been unusually fortunate in planning its development of modern mathematics curricula. Reforms of early university mathematics education were being planned a decade ago in the United States. These created new demands for the modernization of high school curricula. A Commission on Mathematics was established and worked through the mid-1950s under the chairmanship of Professor A. W. Tucker, Princeton University. While this Commission had no official national standing, its report has been widely read and has been immensely influential. (Copies may be obtained from the Educational Testing Service, Princeton, N. J.)

As soon as this report was published, it became clear that at least two steps had to be taken to make any reform in the United States a reality. One was the introduction of suitable text materials, even if they were of an experimental nature. The second was the training of tens of thousands of high school mathematics teachers who had never been exposed to modern mathematics. Here the National Science Foundation came to the aid of the mathematicians. Through grants, amounting to many millions of dollars, the National Science Foundation established means of meeting both of these problems.

First of all, special institutes were established for the retraining of high school mathematics teachers. Each summer thousands of mathematics teachers are enabled to study modern topics in mathematics with all their expenses paid by the Foundation. More recently, the Foundation has enabled mathematics teachers to return to universities for an additional year's study.

The writing of experimental text materials was started by various university groups, notably one at the University of Illinois. More recently, the National Science Foundation made possible the setting up of a national writing group, the School Mathematics Study Group, under the leadership of Professor E. G. Begle, originally of Yale University and now of Stanford University. Over a period of five years more than 100 mathematicians and mathematics teachers have cooperated in the writing of a series of experimental materials. These have been widely tested throughout the United States and have been rewritten until they form both highly acceptable experimental text materials and will form a basis for future textbooks on the subject. (Information about these materials can be obtained from the School Mathematics Study Group, Stanford University, Stanford, California.)

The problem of implementation is made infinitely more complex in the United States than even that noted in the German report, since the final decision on curricula in most cases is neither in the Federal government's hands, nor in the hands of State governments. The latter usually set minimum standards, but the details of curricula are voted on by each individual community. Therefore, before reform is complete, many thousands of local school boards have to be persuaded of the desirability of modernizing their mathematics curricula. On the other hand, this local control also had its advantages in starting wide-scale experimentation. In many states it would have been impossible to get the State governments to approve the new curricula, because of lack of qualified teachers, but individual cities or towns were able to adopt new topics without waiting for State approval. We therefore find a strange situation in the United States, where one may find hundreds of schools with perhaps the most modern mathematics curricula in the world, and at the same time still find thousands of schools that have not even given any thought to the modernization of high school mathematics teaching.

In conclusion, I would like to reiterate a sentiment contained in the German report, namely that it takes at least a generation to complete a major change in the mathematics curriculum. At the rate mathematics is developing, by the time the present reform is completed, we are sure to want a reform of the "modern curriculum".

This is perhaps dramatically illustrated in the United States by some exciting experiments carried out in the last three or four years in teaching modern ideas to students in the first six years of school. For example, in the city of Cleveland, a number of suburban school systems adopted School Mathematics Study Group materials, starting with the 7th year of school, and have developed their own materials for the first six years. They are now facing the very serious problem that by the time their students have studied modern mathematics (in an elementary version) for the first six years, they will find the "frightening" new ideas of the 7th and 8th years much too easy, and hence these schools will find the modernized curricula terribly old-fashioned.

3. *The new curricula*

The most striking feature of the 21 reports is the degree of similarity in the proposals for including new topics of mathematics.

There are four areas of modern mathematics that are recommended by a majority of the reports. These are elementary set theory, an introduction to logic, some topics from modern algebra, and an introduction to probability and statistics. Equally frequent is a mention of the necessity for modernizing the language and conceptual structure of high school mathematics.

Perhaps the most frequently mentioned topic is that of elementary set theory. The concept of a set, as well as the operation of forming unions, intersections, and complements, constitute a common conceptual foundation for all of modern mathematics. It is therefore not surprising that almost all nations favouring any modernization of the high school curriculum have advocated an early introduction to these simple, basic ideas. An attractive feature of this topic is that in a relatively short time a student may be given a feeling of the spirit of modern mathematics without involving him in undue abstraction.

It should, however, be noted that in most cases only an elementary introduction of this topic is recommended. For example, the usual „next" topic in developing set theory is that of cardinality. Only three nations have suggested this as a possible topic for inclusion in the secondary curriculum.

The introduction of elementary symbolic logic may be justified on grounds quite similar to that of the introduction of sets. Indeed, the most elementary structures in the two subjects, Boolean algebra and the propositional calculus, are isomorphic. It is, therefore, not surprising that in several countries these topics are studied more or less simultaneously, exploiting the various possible ways of setting up isomorphisms between the systems.

Of course, logic plays a strange dual role in the mathematics curriculum, in that logical reasoning is an underlying feature of all mathematical arguments, and at the same time modern symbolic logic is an interesting topic in its own right. After many centuries of making free use of logic, without careful examination of its basic principles, the mathematician has turned around and made logic one of the branches of mathematics. It should again be noted that in most cases only very elementary principles of logic have been suggested for study in the high school curriculum.

The status of probability and statistics is entirely different from that of logic and sets. The introduction of these subjects into the high school curriculum is proposed usually on the basis of their inherent attractiveness and importance, rather than their instrumental use in other branches of mathematics. In almost all cases both probability and statistics were advocated, usually closely tied together. I shall follow the convention that under the heading of "probability" a branch of pure mathematics is meant, while "statistics" describes a branch of applied mathematics. If this view is accepted, we must see here both the most widely recommended subject in pure mathematics and the only widely recommended subject in applied mathematics, for inclusion in high school education.

I would like to suggest that the extent to which probability theory is to be taught in high school should be one of the topics of discussion following this report to the Congress. Probability theory recommends itself as a very attractive branch of pure mathematics because it is so easy to give examples, from everday experience, involving probabilistic computations. Therefore, the student is challenged to combine mathematical rigor and intuition.

However one may consider introducing probability theory from a purely classical point of view, in which one deals with equally likely events and defines probability simply as a ratio of favourable outcomes to total number of outcomes. In this case, probability problems reduce to problems of counting or combinatorics. There is no doubt that such simple combinatorial problems are well within the grasp of the average high school student and, indeed, such topics have long been included in high school algebra courses. In many of the reports sent to me it was not clear whether the probability theory advocated goes beyond such elementary computations.

To capture any of the spirit of modern probability theory, it is necessary to introduce the concept of a measure space and to define probabilities of various events in terms of measures of subsets. While anything like a full treatment of measure theory is much too difficult for high school students, a number of experiments have shown the possibility of doing this for discrete situations, or even more restricted, for finite sets. Since the normal problems familiar to high school students deal only with a finite number of possible outcomes, this formulation of the foundations of probability theory corresponds particularly closely to the students' every-day experience. Recommendations for such a very elementary treatment of probabilistic measure theory are contained in four reports.

While a majority of reports contained a suggestion that some topics from modern algebra should be chosen, there was considerably less agreement as to what this choice should be. Basically, there seems to be a split between the advocates of teaching topics from algebraic systems (groups, rings, and fields) and those who advocate linear algebra. In a few cases, both types of topics were suggested, but usually the lack of time in high school curricula prevents the introduction of a very sizable amount of modern algebra.

It seems to me that the motivation for these two types of topics have many common features. The introduction, on an axiomatic basis, of any modern algebra has the very healthy feature of removing the common misconception that axiomatics is somewhat closely tied with geometry. I recall once having a student who told me that, in his experience, the difference between algebra and geometry was that "in geometry you proved things, while in algebra somebody just told you what to do". Certainly, this objective can be equally well achieved by introducing as one's basis axiomatic system either that of a group or that of a vector space.

In addition to this, either linear algebra or algebraic systems have the advantage of giving deeper insight into certain structures known to the students for other reasons. Linear algebra, of course, has many applications to geometry, while algebraic structures arise as generalizations of one's experience with numbers.

The usual argument given for the introduction of groups, rings, and fields is that this is the only way one can bring about a true understanding of the nature of our number system. Attempts to prove to the student simple rules, such as those governing the operations with fractions, often fail because both the basic assumptions and the results to be proven are too familiar to the student. However, by moving to an abstract axiomatic system, the student is forced to abandon his intuition and rely on mathematical rigor in his proof.

It may certainly be said, if one wishes to introduce one example of an axiomatic system in modern algebra, that the simplest and most universally useful one is that for a group. It also has the attractive feature that, in addition to being applicable to many groups of numbers well known to the student, one can introduce such simple and interesting examples as the symmetries of a simple geometric object (e.g., a square).

A study of vector spaces, of course, is much more difficult than the study of a simple system such as a group. I have not seen any suggestion of studying vector spaces over an arbitrary field. However, there were a number of suggestions for studying a vector space over the real numbers. Here much of the difficulty is removed by relying on the student's intuitive understanding of the underlying field. Presumably, the major motivation for this line of inquiry is that it helps to clarify much of what the student was forced to learn before. For example, it can be used to give new insight into the meaning of the solutions of simultaneous equations. Equally important, of course, are the numerous applications of linear algebra to geometry. While geometry can be used to motivate linear algebra, linear algebra, in turn can be used to make the nature of geometric transformations more clearly understood.

I must now mention a few topics which occurred occasionally amongst the recommendations, though these seem to be topics not nearly so widely accepted. These include some modern topics in geometry, the study of equivalence and order relations, cardinal numbers, and an introduction to elementary topology. There were also scattered mentions of applications, but this is a topic to which I wish to return later.

There seems to be general agreement that the teaching of high school geometry must be modernized, but there is a certain lack of ideas as to how this should be achieved. I recall the detailed debate at the 1958 International Congress on this particular topic, and I am under the impression that this problem is still far from settled.

For example, the School Mathematics Study Group in the United States wrote single textbooks for each of six years for junior high school and high school mathematics. However, in the case of the tenth year, there are already two different versions of geometry available, and there may very well be a third version. This is a clear-cut indication of the lack of agreement amongst leading mathematicians in the United States as to the "right" way of teaching geometry.

The most constructive suggestions on this topic seem to be contained in the report from Germany, and I refer the reader to the excellent bibliography contained in the appendix. I share the astonishment expressed by the German reporter that high school geometry has remained so terribly tradition-bound, even in the face of many changes in the teaching of algebra, and the introduction of more advanced topics. We must choose between a 2,000-year-old tradition of teaching synthetic geometry in the manner of Euclid, or of destroying the "purity" of geometry by the introduction of algebraic ideas. Of course, Felix Klein established a very important trend in Germany, which spread throughout the world, to attempt to build a classification of geometries by means of the transformations which leave certain geometric properties invariant. This points to the importance of the study of geometric transformations, even within high school geometry. There is also an increasing tendency to introduce metric ideas early into synthetic geometry and in many countries even an introduction to analytic geometry is part of the first year's geometry course.

The introduction of vectors is quite generally advocated. In Germany vectors are introduced in the context of metric (as opposed to affine) geometry. However, this does not mean that vectors are tied to analytic geometry, since vector methods are used as a substitute for the introduction of a coordinate system. This approach is particularly useful in bringing out the analogy between the geometries of two, three, and more dimensions.

A conference sponsored by ICMI at Aarhus, in Denmark, in 1960, advocated the development of a "pure" vector geometry, in which affine geometry is built up in terms of vector ideas. While the concept of vectors free of coordinate systems may be somewhat more difficult for the beginning student to understand, many geometric proofs actually become much simpler if vectors are treated as coordinate-free. For example, this is by far the easiest way to prove that medians of a triangle meet at one point and divide each other in a 2:1 ratio.

While there are still many advocates of treating a full axiomatic system of Euclidean geometry purely synthetically, it is becoming increasingly clear that one must either "cheat" or demand more of the student than can be expected of him in his high school years. Even Euclid's original axiom system is a great deal more complex than is ideal for the high school student's first introduction to axiomatic mathematics. In addition, it is well known that Euclid in many places substituted intuition or the drawing of a diagram for mathematical rigor. Indeed, many of Euclid's propositions do not follow from his axioms. While several outstandingly fine axiom systems have been constructed that make Euclidean synthetic geometry rigorous (notably the system by Hilbert), these require a degree of mathematical maturity not to be expected of the secondary school student.

The report from Israel feels that the axiomatic treatment of geometry in high school is as unrealistic as using Peano's postulates in elementary school. The report from the United States, in contrast, advocates that certain *segments* of Euclidean geometry be taught rigorously, to give the student experience in proving theorems from axioms, but that the gaps in between be filled in by a more intuitive presentation, in which the emphasis should be in teaching students the "facts of geometry". An alternative to this is the much heavier reliance on the properties of real numbers to fill in gaps in Euclid's axiom system.

Three reports advocated the inclusion of non-Euclidean geometry as part of the first treatment of Euclid. The argument for this is similar to the argument for teaching algebraic systems to improve the students' understanding of number systems. That is, if the student is forced to reason in a geometric framework other than the one he is used to, he is more likely to understand the power of the deductive system and to appreciate proofs he has seen in Euclidean geometry. I should like to add a plea that, even in courses where no actual non-Euclidean geometry is taught, the student should at least be informed that such geometries do exist, and perhaps a day or two be spent discussing them. It seems to me to be a major cultural crime of most mathematical educational systems that 130 years after the invention on non-Euclidean geometry, most students (and I am afraid many teachers) are not aware of the possibility of a non-Euclidean geometry. Indeed, the statement that our universe is only approximately Euclidean, according to relativity theory — it may both in the small and the large be non-Euclidean — comes as a great shock to many pedagogues.

A frequently mentioned topic is a brief study of relations in general, with special emphasis on equivalence relations and order relations. The justification for such fundamental concepts is the same as for a brief study of sets and of symbolic logic; once these concepts are introduced, they can be used again and again to clarify later topics.

Three reports suggested the inclusion of a systematic study of cardinal numbers. I must say that this suggestion both delights me and surprises me. It delights me in that I have always been critical of university education in the United States, in that most students are supposed to learn the facts about infinite cardinals entirely on their own, since these topics are rarely explicitly taught in courses. The suggestion surprised me because I had felt that this topic was too difficult for high school curricula. If various countries succeed in this experiment, I think it would be most useful if the results were widely publicized.

Suggestions of a brief introduction to topology are contained in four reports. The French report proposes that an intuitive notion of neighbourhoods be given to students and on this one should base the concept of the convergence of a sequence (or the failure of convergence) and that these ideas should be used to lead in a natural way to the concept of limits and continuity. These can in turn be used to explain such geometric ideas as that of a tangent or of an asymptote. Germany and Israel make similar suggestions.

A more ambitious program is outlined in the Polish report. The proposal is that most of the treatment be restricted to the topology of Euclidean space of one, two, and three dimensions. Starting with these well-known spaces, the concept of a metric space should be developed, and, in turn, illustrated on such examples as n-dimensional space, the space of continuous functions, and Hilbert space. The Polish program would start with the same concepts as mentioned above from the French report. However, by limiting itself to more concrete examples, it proposes to go considerably homeomorphism, and continuous mappings would be discussed. More concretely, it is suggested that discussions without proofs should be given of the Jordan-curve theorem, classification of polyhedral surfaces, and some examples of non-orientability of surfaces. The unit would terminate with a discussion of Euler's theorem.

Your reporter would like to add his support to this suggestion, even though it may sound quite extreme. While these topics may be too difficult for the average high school student, I know from personal experience that the really bright student, in his last year of high school, is fascinated by elementary topological ideas. Such a unit should be entirely practical as long as it is closely tied to concrete examples familiar to the student.

Most of the reports contained frequent mentions of traditional topics whose teaching would be improved by the adoption of a more modern point of view. As one example, I shall use a unit discussed in the report from the United States. This is the treatment of equations, simultaneous and inequalities. An equation or inequality is treated as an "open sentence". That is, it is a mathematical assertion which in itself is neither true nor false, but becomes true or false when its variables are replaced by names of numbers or points (or more abstract objects, in advanced subjects). Therefore, the solution of an equation is the search for the set for which the assertion is true. This set is commonly referred to as the "truth set" or the "solution set".

Thinking of solutions of equations as sets has the advantage that a student is more likely to think of the possibilities of the solution having more than one element in it or, for that matter, being the empty set. Simultaneous equations may be thought of as conjunctions of several open sentences; hence their solution consists of the intersection of the individual truth sets. This point of view makes it much easier to explain the usual algorithms for solving of simultaneous equations. The attempt in any such algorithm is to replace a set of sentences by an equivalent set, i.e., one having the same truth set, but the latter being of a form in which the nature of the solution is obvious. The approach also had the advantage that equations and inequalities may be treated in exactly the same manner. The graphing of equations and inequalities, then, simply becomes a matter of graphical representation of truth sets. In this case, the meaning of "intersection" of solution sets becomes particularly clear.

4. *Applications of mathematics*

It is painfully clear, in reading the 21 national reports, that relatively little attention has been given by our reformers to the teaching of applications of mathematics. The only notable exception to this is the inclusion of statistics in a majority of the recommendations. Aside from this, only scattered suggestions are made, none of them occurring in more than two reports. Indeed, some reporters have specifically complained that, while an enormous effort has been made in their nations to improve the teaching of pure mathematics, the topic of applied mathematics has apparently been forgotten. I would like to propose to ICMI that a study of the teaching of applications of mathematics should receive high priority in its studies of the next four-year period.

Aside from statistics, three types of applications have been mentioned. One is applications of mathematics to physics. I presume it differs greatly from country to country as to whether topics such as mechanics are included in the mathematics curriculum or are treated in separate physics courses.

A second area that was mentioned twice was that there are great possibilities in the future of improving the teaching of mathematics by making free use of computing machines. Of course, in the immediate future this may not be practical until high-speed computers are available in large enough numbers for high school students to be able to give sufficient time on them.

A third area mentioned was linear programming. This particular topic has the attraction that it ties up nicely with linear algebra and therefore can reinforce the teaching of a quite modern topic of abstract mathematics. It also lends itself to good numerical problems which are both interesting and will exercise the student's ability in the solving of equations. But, above all, it may be the only example the student will see of a genuine application to the social sciences.

The philosophy of teaching applied mathematics is particularly well described in the report from the Netherlands.

"It is an urgent problem whether secondary education must restrict itself to pure mathematics. Applications gain more and more momentum in the social system. If these applications were only operational, one could ask whether they should be taught at all in high schools. Teaching applied mathematics, however, implies developing new habits of thinking, which in many cases differ from those in abstract mathematics. For instance, in statistics it is difficult to acquire operational skill as long as one has not really and independently understood the fundamental notions".

5. *Further observations*

Perhaps the major motivation for teaching modern mathematics, or mathematics in a modern spirit in high school, is to prepare the student for his university experience. The need for this is particularly well brought out in a quotation from the French report form Professor Lichnerowicz. The quotation (in translation) reads: "The classical teaching of our lycées in a large measure conditions our students to a certain conception of mathematics, a conception which is . . . derived from the Greeks, and . . . from the experience of mathematicians of the middle of the nineteenth century . . . At the university, the students suddenly encounter the spirit of contemporary mathematics, a painful shock . . . The student must totally 'recondition' himself . . . and this is translated by an expression which I personally have often heard: 'What you are teaching is no longer mathematics' . . ."

I am sure that many of us can testify to the same experience. Let us now examine a few pedagogical problems.

The Netherlands report recommends that "stress should be laid on thinking mathematically and more value attached to this ability than to knowledge of a variety of less important facts." If this philosophy is adopted, then presumably the exact choice of topics is not nearly as significant as the manner in which they are presented in the high school.

An important pedagogical idea is expressed in the Portugese report: "For this introduction (of modern mathematics) it would be essential to bring out many concrete examples, well known and quite suggestive, as well as amusing, and one would be careful not to introduce formalism until one was sure that the student had grasped the ideas behind them."

One question that arises in the introduction of new topics is what topics are reduced to make room for the inclusion of new ideas. By far the most frequently mentioned topics were a reduction in the amount of time spent on synthetic geometry, a considerable reduction of trigonometry, especially the emphasis on triangle solving, a reduction of solid geometry, possibly by incorporating it into the first course in geometry, and a reduction in some of the traditional and not very practical numerical methods included in algebra courses.

A pedagogical question on which there seems to be considerable disagreement is the extent to which high school mathematics should be axiomatized. I found several recommendations that there should be a substantial extension of the body of axiomatics in high school, or even that axiomatic systems, as such, should be studied. On the other hand, there were about an equal number of objections to excessive use of axiomatization in the modernized curricula. For example, "The enrichment of the syllabus by the insertion of interesting examples of modern elements of mathematics is to be encouraged, and indeed is bound to happen. But the systematising of teaching in line with axiomatic mathematical theories would lead to a situation contrary to accepted British teaching principles."

A different view concerning axiomatics is shown in the French report: „*Axiomatic Exposition.* A program cannot demand that teaching have an axiomatic character until sufficient scientific experience permits the student to feel its need. Axiomatic procedure is extremely rigid, each step is strictly controlled, appeal to the intuition has no value because the choice of axioms accepts some facts and rejects others just as sympathetic to our intuition. If the construction succeeds and gives what our experience of the question expected, if one has more or less demonstrated the independence of the axioms and the categorical quality of their set, one sees that the choice was good. But who will believe that such a choice can be made without fumbling? and different axiomatizations are valid. It is impossible to set them forth without dogmatism, without appealing to the authority of the teacher who is able to show only to the end that the work is valid.

"In secondary school teaching one can only try to come to the conclusion that axiomatics are doubtless possible and desirable in mathematics. In terminal classes, it is recommended to do a few axiomatic expositions at the outset, granting the necessity of afterward accepting a more technical viewpoint. But it is very dangerous to do partical axiomatics, which hide the unity of mathematics even if one doesn't make vicious circles (like using number to axiomatize geometry and geometry to axiomatize the notion of number!).

"However, even if a large place is left to the intuition of the children and the path chosen for exploring the program is flexible and takes account of the spontaneity of the students . . . it is necessary for the teacher to impose an order without which there would be only confusion. This order reflects an underlying axiomatization adopted by the teacher, of which the best pupils can become aware at the end of the school year."

In the historical development of mathematics, it is usually, though by no means always, the case that a certain body of mathematical facts is first discovered, and then one or more people perform the very important task of systematizing this information by specifying a minimal number of axioms and deriving the other facts from these. It is therefore clear both that some acquaintance with axiomatic mathematical systems is an important part of mathematical education, but also that mathematics is something over and above mere development of axioms. Just what the happy compromise is between these two trends may be a topic well worth discussing at the Congress.

The newly developed Danish curriculum provides a very interesting idea — namely, an optional topic to be selected by the high school teacher. The choice of this topic is described as follows:

"Contents, extent and mode of treating the optional subject should be adapted in such a way that the students are not in this field faced with more difficult problems than those arising from the other lessons of mathematics.

"Some examples of the fields from which the optional subjects may be taken: History of mathematics, number theory, matrices and determinants, theory of groups, set theory, Boolean algebra, differential equations, series, probability theory, statistics, theory of games, topology, projective geometry, theory of conics, non-euclidian geometry, geometry of higher dimensions, geometrical constructions, descriptive geometry.

"The optional subject may also be chosen in connection with the corresponding part of the physics course. As examples of suitable subjects may be mentioned: Probability theory and kinetic theory of gases, differential equations and oscillatory circuits. Finally the optional subject may be organized in connection with other subjects than physics, e.g., probability theory and heredity.

"The program for the optional subject will have to be submitted to the inspector of schools for approval.

"The existence of an optional subject in the mathematics curriculum is new in Denmark. This subject will have such an extent that a couple of months in grades 11 or 12 will be occupied by it. Of course, both modern and classical subjects will be chosen, but it is expected that many teachers will choose the theory of probability as their teaching subject. In the list of non-optional subjects probability does occur, but only on a very modest scale. Of course the teacher is free to choose between an axiomatic and a non-axiomatic treatment of probability, but certainly an axiomatic treatment will be used by some teachers. (This will probably be easier to carry through if one restricts oneself to discrete sample spaces.) In this case the pupils will get a very useful impression of a simple axiom system and an example of a mathematical model."

One topic mentioned in a number of reports is the extent to which calculus is included in the secondary school curriculum. I have not specifically discussed this topic since it cannot legitimately come under the heading of "modern mathematics". However, it is clear that there are increasing pressures from physical scientists to teach some units in calculus in our secondary school curricula, and to a great extent this pressure may complete with the demands for modernizing of modern mathematics. Let me simply indicate that at the present time there are vast differences from the majority of countries that teach no calculus at all in the secondary school to the large number of countries that teach a first, more or less intuitive introduction to calculus, to such extreme as the recent experiment in Sweden. A special experimental unit will be taught in that country on differential equations: "This small course consists of linear equations of first order and of second order, with constant coefficients. Proofs of existence and uniqueness are given."

The Hungarian report calls attention to two problems that have caused difficulties in modernizing the high school curriculum: "One is the preparation of the teachers now teaching for the handling of new subjects. Without this, the introduction of such topics cannot succeed. But equally important is the formation of public sentiment, since for the majority of people it is not obvious why their children in high school should learn about problems that their parents may never have heard of in their entire leves. We have to solve these problems simultaneously with the modernization of the curriculum."

I am quite certain that many reporters would heartily support these remarks. There are indications in many reports that major national attempts have been made to modernize the training of existing high school teachers. This is, of course, often a highly painful and difficult experience for adults who have left their universities with the impression that they are prepared to teach mathematics for the rest of their lives, and find themselves forced to return to study what often seems to them strange new ideas.

Speaking for the United States, I may add that the problem of informing parents of high school children is equally critical. In many communities where the schools were happy to modernize the mathematics curricula they ran into unexpected opposition from parents who simply could not understand why modern mathematics should be taught, or even how there could possibly be such a thing as modern mathematics. It is strange that, in an age of fantastically rapid development in mathematical research, perhaps a majority of laymen are under the impression that all new mathematics was done hundreds of years ago.

Most of the reports were from countries with educational systems based on centuries of tradition. I was fortunate in obtaining one report from Africa, which painted a fascinating picture of the problems faced by newly developing nations. I would like to reproduce just one quotation which I found particularly interesting, from the report of Sierra Leone:

"The most important factor in our survey is that in all these areas education has been expanding very, very rapidly within the last ten years. The number of secondary schools has at least doubled in all areas and is still expanding. It is in these new schools that there is the greatest opportunity for introducing modern mathematics.
The teachers in these schools are usually young enthusiasts and, the schools often being in new towns, are sufficiently separated from the older traditional schools to make it possible for experimental work to be carried out without pupils and parents continually comparing the work there with the work being done in other schools."

6. *Conclusions*

It is clear from the reports that many nations have made an excellent start on the modernization of high school mathematics curricula. It is equally clear that much hard work still needs to be done.

There seems to be a fairly general agreement that some basic concepts from set theory and logic should be introduced, that geometry should be modernized, that some elements of modern algebra be introduced, and that probability and statistics are suitable for high school teaching. Even more important is the general agreement that much of traditional mathematics should be taught from a modern point of view. However, as far as the details of these recommendations are concerned, there is considerable disagreement.

The two greatest difficulties blocking progress are the critical shortage of qualified teachers, and the lack of suitable text materials. The former problem has been attacked in a few countries by running special courses for high school teachers whose training was mostly traditional. The latter is being solved by the writing of many excellent experimental text materials.

I should like to conclude the report by making two specific recommendations to ICMI:

Recommendation 1. That ICMI initiate study on three problems that have arisen out of these various national reports: (1) How can the teaching of applied mathematics in our high schools be modernized? It is clear that this problem has been neglected in the past. (2) To what degree should high school mathematics be axiomatized? There is considerable disagreement in this topic. (3) How and to what degree should probability theory be introduced? While this is the subject most frequently recommended as a major new topic, many pedagogical questions concerning it remain to be answered.

Recommendation 2. That ICMI serve as a clearing house for experimental materials on modernizing high school mathematics. That each national subcommission should be requested to send to ICMI a list of available books and articles, with an indication of how they can be obtained, and that this list be kept up to date by ICMI and circulated to the national commissions. This could expedite planning and eliminate unnecessary duplication.

APPENDIX

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CONNECTIONS BETWEEN ARITHMETIC AND ALGEBRA IN THE MATHEMATICAL INSTRUCTION OF CHILDREN UP TO THE AGE OF 15[1]

by

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1. This report constitutes a synthesis of the reports on topic Nr 3 submitted by the National Subcommissions of the following eleven countries: Austria, France, the German Federal Republic, Great Britain, Holland, Hungary, Italy, Poland, Sweden, the U.S.A. and Yugoslavia. In these countries compulsory school attendance begins at the age of 6 or 7; thus the subject of the report is the teaching of arithmetic and algebra in the first eight or nine grades.

The material presented by the reports suggests, above all, one general observation. In all countries extensive studies are being conducted at present, aiming at the revision of school programmes, perfecting the methods of instruction and at the preparation of better textbooks. Studies of this kind are being undertaken by associations of mathematicians, by special working teams in which university professors cooperate with secondary school teachers, and also by individual educators.

Until recently in most countries the school syllabus for arithmetic and algebra deviated very little from the following plan.

The first four or five grades: Natural numbers and zero, decimal notation, the four operations, the metric system of measures.

Grades 5, 6 and sometimes 7: the four operations involving common and decimal fractions, ratio and proportion, percentages, applications in various practical sums.

Starting from grade 7 or 8 (pupils aged 13—14) a new subject — algebra — was taught, comprising the principles of handling rational and later also irrational algebraic expressions, the introduction of relative numbers, and solving linear and then quadratic equations.

School arithmetic and school algebra were thus more or less kept apart. In teaching arithmetic the main objective was to develop

[1]) English translation of the report delivered at the International Congress of Mathematicians, Stockholm, 1962.

skill in numerical calculations and in solving textual problems, at times rather artificial or complicated, by "arithmetical" methods. The teaching of algebra was chiefly concerned with the efficient transformation of algebraic expressions and the solution of equations and their application to problems. Questions of a logical nature played an insignificant role in the process of instruction. Theorems and proofs of theorems existed only in geometry. The unificatory ideas of modern mathematics were entirely unknown.

A similar description of the teaching of arithmetic and algebra in the lower grades of the secondary school was given by Prof. H. F. Fehr in his report at the Edinburgh Congress in 1958 [1]).

Although the teaching approach described above is not yet entirely a thing of the past, in a great many countries considerable changes have lately been introduced, and in many others changes are being planned and discussed. The general trends of the reform are similar everywhere. Namely, attempts are being made to bring school instruction, even in junior grades, closer to present-day mathematics and to its present-day applications, acquainting the pupils gradually with elements of the language of modern mathematics. This involves, for example, introducing as early as possible the simplest notions of the theory of sets and of logic including certain symbols, putting more stress on the structural properties of the sets of numbers under consideration, developing the notion of function as a mapping of one set into another, and its applications. It is considered necessary to pay more attention than before to the conceptual aspect of the material dealt with, and not to be content with developing skill in computations and tranformations.

2. The reports of the National Subcommissions contain a lot of valuable data and interesting opinions on the topic in question; however, they show considerable differences as regards the range of problems treated and the amount of detail in their presentation.

The Dutch Subcommission has presented a very extensive report (over 120 pages) consisting of the papers of 9 authors. These contain a critical survey of a number of important teaching problems, such as: the extension of the number system, the introduction of algebraic notation, the notions of function and relation, etc. Moreover, they give information on the evolution which the Dutch school syllabus has been going through and on methods used in school textbooks. In Holland there is a State Commission, appointed in

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[1]) H. F. Fehr, The Mathematical Education of Youth, Enseignement Mathématique, IIe série, tome V, 1960, pp. 62—78.

1961, to deal with the modernization of mathematical instruction in secondary school. The Dutch report has already appeared in print.

The French report is also very comprehensive. The author discusses the scope of the teaching of arithmetic and algebra in the 7th, 8th and 9th years of school instruction (classes de 5e, 4e et 3e) and presents a scheme for a modern approach of the course in those grades. The scheme is developed in great detail and even contains a collection of very interesting exercises. It is not inconsistent with the school syllabus now obligatory in France since it differs from it mainly in the method of presentation of the prescribed material; actually, the official syllabus envisages the introduction of modern concepts and symbolism on a moderate scale. Similar attempts at modernization are being succesfully undertaken in France by individual teachers.

The German report discusses in detail the syllabus and the methods of teaching arithmetic and algebra in grades from the "Sexta" to the "Tertia", i.e. from the 5th to the 9th year of school instruction. In the German Federal Republic the individual federal states are autonomous in cultural matters; the question of modernizing the school syllabus is treated differently in each of them, but is everywhere on the agenda. In some of the federal states new programmes and new textbooks are already in preparation. The changes to be introduced will probably be moderate; the author outlines the main trends of the proposed reforms.

The Austrian reporter states that the teaching of mathematics in Austria is conducted on well-tried, traditional lines, whose main principles were formulated long ago in the so-called Meran plans. The consciousness of a need for a fresh reform is not yet widespread in Austria. However, the reporter is convinced that Austrian teachers also will soon make attempts to realize new ideas following the experiences of other countries. The report presents the process of teaching arithmetic and algebra to children up to the age of 15 and contains numerous valuable remarks on teaching methods.

The Hungarian reporters analyze in detail the connections between school arithmetic and school algebra, and advocate the removal of the artificial dividing line between these two subjects of instruction: the paper reports on the introduction of algebraic concepts in the lower grades of Hungarian schools. In 1963 a new, reformed plan will begin to operate in Hungary; it attaches great importance to a careful introduction of basic mathematical concepts.

The British report, submitted by the Mathematical Association Teaching Committee, is slightly different in character. In England and Wales secondary schools are not all run according to the same pattern. There are schools for more gifted pupils and schools for less gifted ones, and also schools of an intermediate type. Even in one and the same school pupils are sometimes divided into groups according to their aptitude for mathematics. The organization of teaching is marked by great freedom and the absence of administrative pressure. This results in a striking variety of ways and methods in teaching practice. The report does not concern any particular school or any particular type of school, but deals with certain general matters. The authors point to a definite change in the teacher's views on the teaching of mathematics. It is recognized that the narrow scope of traditional teaching should be broadened and that a more mathematical view of the material taught should be introduced.

The reports of the other National Subcommissions are rather brief, giving a more condensed description of the present state of teaching arithmetic and algebra in those countries and of the reform trends in this field.

In Italian schools the teaching of arithmetic and algebra in lower grades is conducted on traditional lines in an empirical and intuitive manner; the logical coordination of the material plays an insignificant part and appears only fragmentarily: aspects of modern mathematics are hardly involved. The Italian reporters point to a need for moderate reforms and indicate the changes in the content and the methods of teaching which they think desirable. They also express their views on various teaching problems.

The Swedish report criticizes existing school programmes and gives information about the work of the Scandinavian Committee for the Modernization of School Mathematics. The Committee consists of university mathematicians and representatives of secondary schools of the 4 Scandinavian countries. Its main objective is the preparation, in the course of a few years, of modern syllabuses and textbooks for the whole course of school mathematics. Some of those textbooks have already been published; the reporter describes their most essential features.

In Yugoslavia during the eight years of compulsory school instruction algebra is taught after arithmetic. The reporter considers this division the right one since it corresponds best to the pupils gradual development. Modern postulates are followed to a large extend.The report discusses the main methodological problems resulting from this.

In the United States the teaching of mathematics has lately undergone far-reaching changes. Groups of scientists and teachers, formed in various university centres, have taken up the work of preparing modern programmes and writing suitable textbooks of mathematics for secondary schools on a large scale. A large part of this work has already been accomplished. In the sequel I will cite two of the textbooks published by the School Mathematics Study Group (SMSG), namely the "Mathematics for Junior High School" (for grades 7 and 8) and the "First Course of Algebra" (for grade 9). It is worth remarking that the material included in those textbooks is very similar to that proposed for the first stage of secondary education (11—15 years of age) in the syllabus recommended by the Organization for European Economic Cooperation (OEEC).[1]) The U.S. reporter informs that in the U.S.A. instruction by new methods is spreading fast. In 1959/60 the textbooks of the SMSG were used by a large number of teachers and pupils in 45 States, with very good results.

In Poland, the work of modernizing mathematical education was taken up, a few years ago, by the Polish Mathematical Society (PTM). A moderate reform plan prepared by a Commission of the PTM has been accepted as the basis of new official programmes, which will be introduced gradually starting from 1963/64. Suitable textbooks will also be ready by that time.

3. It would be difficult to give an exact answer to the question what is arithmetic and what is algebra in school mathematics. However, the adoption of a clear-cut classification does not seem necessary for the purposes of this report. In the elementary teaching of mathematics the starting point is simple experiments with concrete objects, which lead to the formation of the first concepts regarding numbers and operations. During the successive years of learning the range of numbers known to the pupil extends; new concepts and new symbols are added. The degree of generality and of abstraction increases; school arithmetic undergoes a gradual "algebraization". In dealing with the question of connections between arithmetic and algebra in the lower grades of secondary school the reports of the National Subcommissions have concentrated — quite rightly in my opinion — on expressing views on the scope of concepts and problems which should be considered in those grades, and also on teaching methods, particularly in grades 5—9

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[1]) Synopses for modern secondary school mathematics, OECC, Paris, 1961.

(pupils aged 11—15). The subsequent sections of this report are devoted to the most important of the problems discussed by the national reporters.

4. Algebraic notation. In the process of teaching arithmetic and algebra it is important to familiarise the children early enough with the use of the language of algebra. As has been stressed in the Dutch report, which gives the fullest analysis of the question, this requires more attention than is commonly believed, since the language of algebra differs from the language of every day life and even from the language of rudimentary arithmetic. In teaching practice its principles are often insufficiently explained, and children learn its properties empirically, when their mistakes are pointed out to them. Very often children do not understand the equality sign correctly and write for instance $3 + 7 = 10 + 2 = 12$. To deal with this problem special exercises are needed. It is advisable that children should early get into the habit of using signs $>$ and $<$; it should be explained that expressions in which the signs of equality and inequality appear are arithmetical sentences. Writing them in words we can show that they look exactly like various affirmative sentences of everyday speech. An important syntactic device of the language of algebra are the brackets. These are usually introduced at an early stage in teaching arithmetic (e.g. in Poland starting from grade 3) in exercises where several operations should be performed successively, e.g. $(2 + 3) \cdot 5$; it is agreed to write $2 + 3 \cdot 5$ instead of $2 + (3 \cdot 5)$. Exercises with the use of brackets should be of a more general nature and should illustrate that brackets are used in order to single out certain wholes. That is why it should not be forbidden and branded as an error to use brackets in cases where, according to the accepted convention they are not necessary, e.g. to write $(2 + 3) \cdot 5 = (2 \cdot 5) + (3 \cdot 5)$.

This is connected with another, more important problem, discussed in the Dutch and the Hungarian reports. In elementary arithmetic $2 + 3$ signifies a request: add 3 to 2. In the language of algebra $2 + 3$ signifies the result of addition, i.e. number 5. The understanding of this meaning of arithmetical expressions should be developed through suitable exercises. If this is neglected, the pupils will have difficulty in understanding the meaning of letter expressions, e.g. $a + b$, since how is one to add b to a if it not known what a and b mean? This fundamental conceptual difficulty will not be overcome by means of substitutions $a = 2$, $b = 3$. Before we pass to operations with letter expressions, the pupils should be taught to read correctly (and to construct) various numerical expressions.

The use of letters as symbols which can denote various numbers appears in many countries as early as the fifth year of instruction (France, Germany, Poland). It is restricted at first to writing down the fundamental laws of operations, some geometrical, physical and other formulas, and simple equations, e.g. of the type $2x + 3 = 7$. The British report discusses here in detail the connections between arithmetic and algebra expressed in the process of generalizing arithmetical facts and gives numerous examples of exercises with different degrees of difficulty. It stresses the usefulness of simple transformations, e.g. of the formula for the area of a rectangle:

$$a = l \cdot b, \quad l = \frac{a}{b}, \quad b = \frac{a}{l}.$$

The transformation of letter expressions is systematically taught as a rule, from grade 7 or 8 upwards, when the pupils are already acquainted with relative and fractional numbers. The systematic solving of equations is begun in the same grades.

Several reports give a closer analysis of the problem of introducing letter notation. Three cases of the occurrence of letters are distinguished: a) as general names for numbers (indeterminates), e.g. $a + 1 = 1 + a$, b) as unknowns, e.g. $a + 1 = 2$, c) as variables, e.g. in the function $a \to a^2$; the reports discuss the question of the proper order in which these cases should be tackled in teaching. In the opinion of one of the Dutch reporters the order "first the unknown, and then the indeterminates" has the advantage of making it possible to begin by solving suitable easy problems. On the other hand, the inverse order better emphasizes the essential logical character of letters as subject variables whose values are numbers out of a certain set. The question which is to appear first, the unknown or the variable, cannot be solved in isolation from the overall teaching method adopted. In any case it is suggested that the term "variable" should be avoided for a time, since it might be misleading; its introduction should be put off until it is needed for dealing with functions.

In Yugoslavia the following method is practiced. Simple equations appear already in arithmetic. In the very first algebra lessons letters are introduced as variables and algebraic expressions as functions of those variables. At the beginning simple problems of familiar types are chosen, e.g. concerning buying and selling, motion, etc.; by changing the numerical value of one of the data linear functions of one variable are obtained; the introduction of the more general concept of functions is postponed.

The British report also favours an early introduction of the concept of variable.

My own belief is that the distinction of this or that particular role of letters is not very essential and should rather be avoided. It is important, on the other hand, to divest the letters a, b, . . .x at once of the mysterious quality they often have for beginners. I think that the following teaching procedure, resembling that adopted in the SMSG textbooks, can be recommended.

1. We explain the use of letters as symbols for which we can substitute names of things belonging to a certain set: "The town N", "the pupil X", "the number a" . . . . We explain that in algebra it is usual to say, shortly, "a".

2. We give examples of true propositions and false propositions: "Rome lies in Europe", "Tokyo lies in Europe", "$1 + 4 = 5$", "$1 + 3 = 5$", "$5 > 3$", "$2 > 3$" . . . .

3. We consider expressions (propositional forms) with one variable: "The town N lies in Europe", "$1 + a = 5$", "$b > 3$", "$1 + x = x + 1$", "$y + 1 > y$"; we stress the fact that none of these expressions is either a true or a false proposition: it is only by substituting the names of objects from a suitable set for the letters that we obtain true or false propositions. For such examples we establish suitable "sets of solutions".

4. We consider similar examples with 2 and more variables.

5. We show that true or false propositions can be obtained from propositional forms by means of quantifiers (without introducing yet the term "quantifier" and the corresponding symbols).

"For a certain number a we have $1 + a = 5$". "For every number a and for every number b we have $a + b = b + a$".

It seems to me that formulating the fundamental laws or arithmetic with the use of quantifiers is very advisable if the pupils are to understand their meaning thoroughly. Later on we can agree to omit them for brevity.

6. We form *interrogative* sentences:

"For what a $\quad 1 + a = 5$"? or "For what a $\quad 1 + a > 5$?"

In this way we obtain algebraic *problems*: to solve an equation, to solve an inequation. The letters in question are then called unknowns.

Some educators in Poland suggest that the problem of solving an equation or an inequation be formulated — at least at the beginning — in the following way:

$$1 + a \overset{?}{=} 5, \qquad\qquad 1 + a \overset{?}{>} 5,$$

in order to stress the fact that here a certain question is posed. It seems to me that the suggestion merits attention.

In the system outlined above simple equations, inequations, and identities are introduced at the same time. I think that in teaching mathematics it is advisable to bring related subjects close together and that for the good understanding and mastery of the language of algebra varied and many-sided exercises are needed, just as for the exact understanding of the meaning of a foreign word it is necessary to use it in different contexts.

5. Equations. Solving equations can be regarded as one of the subjects linking up the teaching of arithmetic and algebra. Several reports stress the possibility of beginning them early, without introducing general principles, on the basis of what the pupils know about operations. As early as grade 1 children solve problems of the type $2 + ? = 5$ or $2 \cdot \quad = 6$, making use of the fact that addition and subtraction, as well as multiplication and division, are inverse operations. Later on we can give a more general explanation of inverse operations as those which annul each other. This easily leads to the solution of linear equations like

$$(x + 3) \cdot 7 - 2 = 26,$$

where the pupil is to find the operation inverse to "adding 3, multiplying by 7, subtracting 2". In Hungary equations of this kind are solved as early as grade 5. In a similar way simple linear inequations can be solved (new syllabuses in Poland). This permits the solution by means of equations of various problems usually solved by traditional, often artificial, arithmetical devices. This early use of equations is recommended in some of the reports, e.g. the Austrian one; the Italian reporter, while recognizing the advantages of this method, believes however that arithmetical devices, e.g. the use of propositions, are very instructive and should not be discarded. It would seem best to use both methods — one or the other — according to circumstances.

The systematic treatment of linear equations is undertaken in most countries in grade 8 and is often connected with solving linear inequalities; quadratic equations appear in grade 9. I have not found in the reports any information on how the logical principles of solving equations are explained. As far as I know, many teachers are of the opinion that the concept of equivalent equations and the fundamental theorems on the equivalence of equations are difficult for the pupils. It is suggested that a certain number x_0 should be assumed as the solution of the equation; then by transforming the hypothetical equality we obtain the value of x_0. In this method the checking by substitution of the solution obtained is an indispensable link in the solving process, which must not be omitted. However, if the pupils are trained in this way in solving linear equations for example, they notice after a time that the number x_0 obtained always satisfies the equation; accordingly, they think that the demand of substitution is only the teacher's pedantry. It seems to me that, at a suitable moment, we should explain the thing more fully, asking the question whether the sequence of equalities obtained in solving the equation could be run over in the inverse order. The discussion of this question easily leads to the required theorems on the equivalence of equations. On a higher level of instruction the concept of equivalence will have to be extended to systems of equations and inequalities.

6. Sets. It is generally agreed that set-theoretical concepts can not be dispensed with in a school course designed on modern lines. It is necessary, however, to consider the extent to which they can be dealt with in lower grades. I think that on the elementary level we should only be concerned with teaching the children some elements of the language of the theory of sets; we should then constantly use that language, which will make further study easier. We should not aim at giving the pupils any comprehensive body of knowledge in this field, such as for instance the various formulas of the algebra of sets. It also seems advisable to introduce a very small number of new symbols, keeping to verbal terms. It is true that new symbols in themselves do not present any particular difficulties to the pupils, but to introduce too many new signs is not pedagogically sound. The papers of the National Subcommissions contain certain proposals regarding this point. The French scheme places information on sets at the beginning of grade 5e (7th year of teaching) and connects it with the treatment of the properties of the set of natural numbers. The following concepts are introduced: set, element of a set, equality of sets, inclusion of sets, empty set, complementary sets, intersection of sets, union of sets, and the symbols $\in$, $\subset$, ϕ, $\cap$, $\cup$; the signs $\rightarrow$ and $\leftrightarrow$ for implication and equivalence are also used. Some simple formulas, e.g. $A \cap A = A$, $A \cup B = B \cup A$ etc., are also given. In the next grade the concepts of propositional logic are explained; implication, equivalence, negation, conjunction and disjunction. They are presented with the aid of their set-theoretical interpretation without introducing new symbols.

In Germany a similar programme is envisaged. The suggestion is that it should be realized gradually, the concept of set being introduced as early as grade "Sexta" (the 5th year of teaching) and hence forth constantly used; some textbooks have already followed this plan. The same notions are mentioned in the Dutch report with the addition of the quantifiers "there is" and "for all". The reporter does not indicate in which grades these subjects should be included, but advises approaching them in connection with geometry (e.g. while dealing with loci) and with algebra (the solution of equations and inequations). A very simple and easy presentation of some concepts of the theory of sets and of logic can be found in the SMSG textbook for grades 7, 8 and 9. The number of terms and symbols is there comparatively small; the main stress falls on the concepts of sentence, propositional function (open sentence) and the "truth set" of a propositional function and also on algebraic and other applications.

In Sweden the introduction of simple concepts of the theory of sets is contemplated in grade 7, in Poland — in grade 9.

The following observations occurs to me. When explaining the meaning of the word "set" on the elementary level we usually speak about sets as collections of certain objects and give various examples from every-day life. This works well as long as we deal with finite sets. Passing to infinite sets, we may encounter difficulties. When I talked to a few 12-year old children about the set of natural numbers, one of them said in surprise: "Why, surely there is no set of all natural numbers since we can always have still more numbers!" I think it would be advisable to explain the use of the word "set" in the following way: instead of saying "5 is a natural number" we say "5 belongs to the set of natural numbers", or "5 is an element of the set of natural numbers". This would be more in accordance with the mathematical sense of the concept of set and would pave the way for the future use of the notion of a family of sets.

7. Functions and relations. The introduction of the concept of function is discussed in all the reports. Most of them share the view that preparations for introducing this notion should be made as early as possible, say in grade 5 or even earlier, when dealing with such questions as the dependence of the results of an operation on the given numbers, direct and inverse proportionality (Austria, Hungary) or handling simple statistical tables (Great Britain). The view is expressed (and particularly emphasized by the Yugoslav reporter) that the concept of function should be developed gradually throughout many years of instruction; the general definition need not be given early, but from the very early grades, particularly from grade 5, children should be systematically trained in functional thinking, the teaching material being utilized for this purpose. A

more general definition of function can be given in grade 8. The French reporter introduces functions in connection with algebraical computation in grade 4e (8th year of teaching) using the notation $x \rightarrow f(x)$, also recommended in the German report. The system of coordinates and the geometrical illustration of the simplest functions appear in grade 6 at the earliest (Poland) but usually not earlier than grade 7 or even 8 (France, Germany, Sweden, the SMSG).

In the schools of most countries, it seems, the functions dealt with are mainly, or even exclusively, numerical functions of a numerical argument, illustrated by graphs in the orthogonal system of coordinates. One might feel doubtful, whether the restriction to this particular case is justified. We should rather try first to inculcate in the pupil's minds the modern general idea of function — as suggested by the Italian reporter and one of the Dutch reporters — by considering various simple examples of mapping one set into another. This can be done in a natural manner, easily understood by children, say at the age of 12, by using concrete examples from everyday life. Moreover, we have a chance here of connecting algebra with geometry, where the notion of mapping is particularly useful, both in the intuitive teaching of the lower grades and on the higher level. One of the Dutch reporters remarks that the use of graphs, indispensable in investigating the properties of algebraic expressions, obscures the notion of mapping; that is why it is important that the children should have a previous acquaintance with mappings on a varied material.

The French and the Dutch reports propose the introduction in the teaching plan of certain notions of the theory of binary relations, namely the notions of reflexive, symmetric, antisymmetric and transitive relations, the order relation, the relation of equivalence, equivalence classes and definition by abstraction. These concepts are explained by means of examples of relations occurring at every step in the teaching of mathematics, e.g. equality and inequality of numbers, inclusion of sets, divisibility, parallelism etc., and also by various examples from everyday life, such as diverse family relationships for instance. The properties of relations can be illustrated in a simple, easily understandable way by means of graphs or tables (the Cartesian product).

However, a question arises in this connection. Personnaly I believe that elements of the theory of relations are more important than many topics of traditional school mathematics. If we teach the children to notice relations and formulate their properties, we shall do more for their education in mathematics than we would do for instance by training them in solving all sorts of involved equations. But should general concepts be introduced and handled at this elementary stage? Would it not suffice to say, for instance, that all congruent segments are of the same length, without explaining that congruence of segments is a relation of equivalence, owing to which the set of all segments can be divided into classes of congruent segments? At the time that schoolchildern make their first acquaintance with the various examples of relations and acquire a certain crude knowledge of mathematics, it is perhaps too early for generalizations, which, a few years later, will prove extremely interesting and instructive to them. It seems to me that Van Hiele's theory of levels of thinking, which is mentioned in the Dutch report, might be applied here; on the lower level relations would constitute "operational matter", on the higher level they would become "subject matter".

8. The concept of number.

a) Natural numbers. The formal theory of natural numbers cannot of course be a subject of teaching for children below the age of 15. During the first few years of school instruction the pupils gain a certain knowledge of natural numbers and operations of them in an empirical and inductive way. In grades 5, 6 or 7 there is usually a systematic revision and summing up of this knowledge on a slightly higher plane. The following points seem to be regarded as essential: a thorough grounding in the fundamental laws of addition and multiplication including the properties of 0 and 1, the introduction of subtraction and division as inverse operations to addition and multiplication, the explanation of the algorithms of operations in the decimal system of numeration with the aid of the fundamental laws of operations. As a preparation for the introduction of fractional numbers, divisibility of numbers, decomposition into prime factors, and finding the greatest common divisor and the least common multiple of numbers are usually discussed. The fundamental properties of the relations $>$, $<$ and $\neq$ should also be formulated here. The French report introduces also (on the level of grade 7) the relations $\leqq$, $\geqq$; the Italian reporter remarks that they should be postponed until later since pupils in lower grades find it difficult to understand the alternative which occurs here; the relations of being greater and smaller are sufficient at this stage. The SMSG textbook introduces the relation of order in the wider sense and the symbols $\geqq$, $\leqq$, $\ngtr$, $\nless$ at the beginning of grade 9.

An instructive topic in arithmetic is to represent numbers and perform operations in systems of numeration other than the decimal, particularly in the binary system. It contributes to a better understanding of the difference between a number and its symbol and of the laws of operations. This topic is proposed in the French and the Dutch reports and in the SMSG for the 7th year of instruction. In the opinion of the Italian reporter, the subject in question, if skilfully treated by the teacher, will be a kind of interesting game for the pupils, provided it is demonstrated early enough. The Swedish reporter also regards the handling of various systems of numeration as useful, but rather for the more gifted pupils. In Germany the subject is considered suitable for the 5th year (the "Sexta"), particular attention being paid to the binary system and its applications to electronic machines.

A more abstract topic is represented by the operations in the set of remainder classes (mod n). Considering these might be of considerable importance as a step towards the formation of general algebraic concepts. When defining operations on remainder classes, i.e. on objects which are not numbers, and discovering their properties, we prepare the ground for the general concepts of operation and of algebraical structures. It remains to consider whether this is suitable for the level of teaching under discussion here. In the French scheme this topic is worked out in the 7th year of teaching (cl. de 5e) and then supplemented in the next two grades, where it is connected with the solution of diophantic equations of the 1st degree. The U.S.A. reporter mentions experiments made in grades 7 and 8 with surprising success. The Italian reporter concludes from his own experiences that handling remainder classes is entirely within the grasp of children aged 12 or 13. The author of the present report has also made successful experiments with 12 years olds, but of more then average intelligence. The topic seems suitable for slightly older children, say the 14-year-olds.

The most important property of natural numbers, the principle of mathematical induction, can hardly be taught with advantage to children under 15. However, experiments with numbers might be envisaged which would give an intuitive background for the induction principle. This idea seems to underlie the handling of natural numbers in the "Sexta", described in the German report.

b) Rational numbers. In the construction of the system of rational numbers the teaching of fractions in most countries precedes the introduction of negative numbers. Some of the reports support this practice with a number of arguments. The Austrian reporter considers an early introduction of negative numbers out of the question; in parctical computations children fairly early encounter simple fractions, such as $\frac{1}{2}$, $\frac{1}{4}$ or $\frac{1}{8}$, whereas there is no practical necessity to bring in negative numbers. Their introduction is dictated by theoretical reasons and can take place later, which, moreover, accords with the historical order. The Yugoslav reporter is of the opinion that negative numbers require a higher degree of abstraction than fractions, since they involve the introduction of the new concept of signed numbers. The British reporter regards relative numbers as a rather difficult subject for a great many children under the age of 15. The Italian reporter expresses the view that psychological difficulties are not greater in the case of relative numbers than they are in the case of fractions, but, on the other hand, children get used to fractions at a very early stage. From the point of view of systematics the construction of the ring of integers should precede the formation of the field of rational numbers; for pedagogical reasons the inverse order is better.

Some of the reports represent a different standpoint. In the French plan (classe de 5e) and in the German plan (the Sexta), and also in certain Dutch textbooks, negative numbers precede fractions. The reporters hold the view that negative numbers are not more difficult than fractions, and in certain respects even easier. They have numerous connections with real life (temperature, level, reckoning of time etc.). They can be introduced in a natural manner, e.g. by counting backwards: 3, 2, 1, 0, -1. -2, ... In mapping numbers 0, 1, 2 ... on the straight line the pupils themselves hit upon the idea of placing on the other side of the zero point the same numbers marked in some special way. One of the Dutch reporters draws attention to the fact that in the formal theory of rational numbers two steps of abstraction are performed: treating pairs of integers as new objects to be operated on and then forming classes of equivalence in the set of those pairs. From this point of view the formal theory of rational numbers is more difficult than, for instance, the theory of complex numbers or the theory of remainder classes, where we have only one of those abstractions to deal with. That is why the formal theory of rational numbers must not be taught too early. It might be observed that conceptual difficulties of this sort occur even on the very elementary level of teaching fractions. The new numbers that are being introduced must be expressed by means of pairs of integers, different pairs representing one and the same number. It is true that the pupil learns these facts on concrete examples, but the situation still seems to be more complicated in teaching fractions than it is in teaching negative numbers.

I myself should be inclined to favour the following procedure. Fractional numbers and negative numbers are taught in two stages. On the level of, say grade 4, (9—10 years of age) fractions are treated "monographically", i.e. children are introduced to certain fractions with small denominators related to practical everyday situations, and possibly also with the simplest cases of adding and subtracting such fractions, e.g. $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$. Not much later negative numbers are introduced in a similar manner, by using, for instance, the profit-loss interpretation. The second stage — beginning in grade 6 or 7 — comprises a systematic treatment of rational numbers. Here there are weighty reasons for keeping to the order: natural numbers — integers — rational numbers.

Nearly all the reports discuss the methods of introducing negative numbers and fractional numbers. Some of them push to the foreground the explanation of the concrete sense of negative numbers by using the well-known examples of opposed magnitudes or directions, and that of fractions by considering measurements or division of a whole into parts. Other reports recommend a more algebraic starting point: new numbers are introduced to make a subtraction or a division possible and they are defined as differences or quotients of natural numbers, or, which amount to the same, as solutions of the equations $a + x = b$, $a x = b$. It must be observed here that both explanations are indispensable on this level and supplement each other. One might only discuss the order in which they should be given; this, in my opinion, depends on the pedagogical inclinations of the teacher. In the latter method a certain algebraic problem is posed, and in the former a concrete interpretation is obtained of the solution of that problem, which, for pupils on this level, plays the part of the proof of existence of the solution (as has been stressed in the Dutch report). Formal constructions proving existence in the mathematical sense would be premature here.

The question how to motivate the definition of operations on integers or rational numbers has been discussed by many educators. Some of them suggest that the operations should be introduced in connection with a certain concrete application. E.g. multiplication of a number represented by a vector by a negative number might be defined as stretching and change of direction. The rules of operations are very easily obtained in this way but the method has a drawback: the pupil does not really understand why such and such a procedure is called, say, multiplication. It seems better to derive the rules of operations from the principle of conservation of formal laws, which many of the reports give in several variants. Afterwards comes the explanation of the interpretation of operations, which makes it possible to apply them in practice.

Some textbooks stress the conventional character of the definitions of operations, pointing out that their motivation by quoting the laws of operations is by no means a proof but only a plausible argument. This standpoint is strongly criticized in the Dutch report. The conventions adopted are not arbitrary at all and their above-mentioned motivations have a clear mathematical sense: they state that the definitions must be such as they are and no other if the operations are to have the required properties. Thus they are — as regards extending the system of numbers — proofs of the unicity of solutions. Granting that a mathematical justification of the existence of solutions must be disregarded on this level, the establishment of their unicity can easily be carried out.

The teaching of the properties of rational numbers can be brought closer to modern algebra by stressing the structural properties of the sets of numbers which the pupils get to know. It is a question not so much of introducing general definitions as of listing the properties characterizing groups, rings and fields on the example of, say, the multiplicative group of positive rational numbers, the ring of integers, the field of rational numbers, possibly also the ring of remainder classes (mod n) and the field of remainder classes (mod p). Later there will be plenty of opportunities for discovering the same properties in other cases: groups of elementary geometrical transformations, the ring of polynomials, the field of rational functions, the field of real numbers and others. Instead of doing traditional exercises in rationalizing the denominator of a fraction, it would be more instructive to ascertain that, for instance, numbers $a + b\sqrt{2}$ (a, b — rational numbers) form a field.

Decimal fractions are usually treated as a particular case of common fractions for which a convenient notation is introduced. The German reporter thinks, however, that in the present day, in view of the general use of the metric system of measures, computation in common fractions does not play the part it used to play formerly; therefore we should first introduce operations on decimal fractions, which are easily explained and applicable to practical problems. In Austria the teaching of decimal fractions immediately after natural numbers is a long-established school tradition. In the new textbooks which are now in preparation in Sweden decimal fractions precede common ones. The arguments for introducing decimal fractions first do not seem very conclusive to me. I am afraid that practical advantages are here achieved at the cost of a conceptual complication, since one more extension of the system of numbers is then necessary.

The set of decimal fractions provides one more example of a ring which is embedded in the field of rational numbers. It is a dense set in this field, which can be shown to the pupils by determining the decimal approximations of rational numbers. Hence we can pass in a natural way to the representation of rational numbers by means of periodic fractions and show that also vice versa: to every periodic fraction there corresponds a rational number. The argument recommended by some of the reports, namely that if $x = 0{,}333\ldots$ then $10x = 3{,}33\ldots = 3 + x$, whence $x = 1/3$, seems to me insufficient, since we assume in it what is to be proved; consequently, a verification of the result is necessary. It would be better to do without this artifice and give a proof based, for instance, on investigating suitable cases or division.

c) Real numbers. The first elementary data concerning irrational numbers usually appear in grade 9, and sometimes even in grade 8. This question is mentioned in a few reports only. Some of the reporters (Austria, Italy, Yugoslavia) stress the impossibility of a more exact treatment of real numbers on this level. In many countries irrational numbers are apparently mentioned only in connection with roots of rational numbers. It is usually stated that the equation $x^2 = 2$ for example has no rational solutions, and yet the length of the diagonal of the unity square satisfied this equation. Hence it is concluded that there are numbers which are not rational. The properties of operations on real numbers are assumed without any explanation as regards the sense of those operations, and then various transformations of irrational expressions are introduced. In addition to that, the pupils are given the traditional algorithm of determining the approximate value of the square root of a natural number N, which is not very instructive; it could profitably be replaced by finding the terms of the sequence:

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{N}{a_n}\right)$$

for instance, as suggested by the SMSG.

I doubt whether the above-mentioned traditional method of approaching irrational numbers is methodically sound. It seems to me that even on this level it would be possible to give the pupils a better idea of the set of real numbers, and that a suitable method would be to define irrational numbers by means of non-periodic decimal fractions — a suggestion made by one of the Dutch reporters and by the French and Polish reporters.

An example of very easy lessons conducted by the above-mentioned method is found in the SMSG textbook for grade 8. They are restricted, however, to the explanation of the concept of real numbers and to their mapping on the straight line. I think that one might go further and give the pupils the definitions of operations on real numbers drawing attention to the laws of commutativity, associativity and distributivity, which can be prepared by considering operations on periodic fractions first.

9. In this report I have tried to present the views of the national reporters on several important problems of teaching arithmetic and the beginnings of algebra. It has not been possible to make use of all the abundant material contained in the reports. Thus for instance it is only to a very small extent that I have been able to take advantage of the wealth of information offered by the many-sided Dutch report. I have also had to pass over a lot of valuable observations in the other reports. I hope that the reports will all appear in print.

As I mentioned at the beginning of this report, the teaching of mathematics is now going through a period of dynamic progress, its general trend being to stress the development of mathematical thinking in the pupils as a preparation for the understanding of present-day scientific ideas and their applications. This does not mean at all that we should neglect training the pupils in arithmetical skills, which obviously are as necessary as ever. On the contrary, we believe that by improving the conceptual side of our teaching we shall be able to obtain efficiency in those skills in a more economical way. The U.S.A. reporter mentions that, according to the tests conducted by the Minnesota National Laboratory, pupils in grades 7—12 who have been taught by the SMSG method have not proved less efficient in computations than those trained in the conventional way.

To-day we still have too small a store of practical school experience to be able to form a critical estimate of the reform introduced. Year by year, however, this store will increase. It would be well, I think, if we could discuss our experiences at the next general meeting of the ICMI.

EDUCATION OF THE TEACHERS FOR THE VARIOUS LEVELS OF MATHEMATICAL INSTRUCTION

by

Kay Piene

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Some time ago, I met an old friend, a mathematics teacher. Years ago we were students together at the university. He is now a very capable teacher. He said: "I see you advocate a new program in mathematics instruction in schools". "Yes", I said. "But if this program is introduced in our schools, I can not teach mathematics any more."

I think this sad story tells us how important the *training* of our mathematics teachers is.

When I was given the honoured task of preparing a report on training of mathematics teachers at this conference, I realized that I had to base my comments on information from different countries represented in I.C.M.I. On the other hand, I also realized that information on *how* teachers are trained *today* is not what we want. We are certainly more interested in knowledge of how — according to experts in these countries — this training *ought* to be organized.

I sent a questionnaire to the different national subcommissions of I.C.M.I. and got answers from around a dozen countries. Not much, but many of them were long and thorough and have given me many ideas, even if the "ideal, future side" in the answers could have been stressed more. Besides I have got valuable information from other sources. By chance this summer Stockholm has had international teacher congresses arranged by WCOTP and FIPESO (World Confederation of Organizations of the Teaching Profession and Fédération Internationale des Professeurs de l'Enseignement Secondaire Officiel). One of the themes discussed here was The Training of Secondary Teachers. I have read and used the reports from the different countries, the summary report and the recommendations from the congress.

What I am going to say now is not the union, or not the intersection of all elements in the answer-sets. I alone am responsible, even if I, of course, am influenced by the answers.

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[1]) Report delivered at the International Congress of Mathematicians, Stockholm 1962.

First some general remarks:

In some countries — Sweden is one — the word mathematics is a word covering arithmetic and mathematics in the older meaning of the word. $7+8$ or $7 \cdot 8$ is here parts of mathematics. I prefer to use both words: *mathematics* when we have proofs and letters besides numbers, *arithmetic* when we have numbers alone and rules for those operations we define for them. (I am tempted in the old saying: „Die Franzosen sagen pain, aber es ist doch Brot" to substitute for the placeholders: Swedes, mathematics and arithmetic).

A mathematical teacher is far from a unique concept. We must know the type of school where he is teaching.

In most countries we find a school system with different levels. First, a primary or elementary school level with arithmetic, but where mathematics may start. Looking back we find this tendency: mathematics is moving down and down. What 100 years ago was taught at universities, we now find in secondary schools.

In accordance with these principles we now find in many countries (real) mathematics starting already in primary schools.

All future primary school teachers today get some instruction in mathematics (and arithmetic) during their training, but this situation will put new demands on them.

In the first years of the primary school, we must have teachers who teach *all* subjects, but later on we must specialize. Not every primary teacher is able to teach mathematics, even on this level. I therefore think Denmark has found a good solution, having an *elective* subject in its training program for primary school teachers. Those teachers who like mathematics can take it. They have also developed a good plan for mathematics in 3 parts: 1) parts of the high (secondary) school curriculum, 2) "professional insight and deeper understanding" (logic, set theory etc.), 3) deeper treatment of some chapters from the primary school program.

But I am not going to discuss the training primary school teachers any further. We return to our model of the school system around in the countries.

The secondary level mostly first has a more general part, then a more specialized (gymnasium, lycee, college etc.) leading to college or university, in the same time giving the highest general education in that country.

The upper secondary school again in most countries is divided in branches — humanistics, classical, modern, natural science, mathematical, commercial, technical, etc., with different demands in mathematics.

To simplify, I assume this model of a secondary school:

1) a first undivided level (11 à 12 — 15 years)
2) a second level (15 — 18 à 19 years)
   a) with a branch specializing in mathematics
   b) and other branches having mathematics as a less important subject.

It should be quite clear that these three school levels must or can have different mathematics teachers.

Our final aim is to give all countries *"good" mathematics teachers.* I do not think it is easy to define a "good" teacher — you cannot find a definition accepted by everybody. On the other hand, it is not an undefined concept. We have at least some ideas of what we mean, we know that some teachers are better than others. — It is therefore also impossible to give *sufficient* conditions for the training and education of a good teacher. But we can give some, more or less *necessary* conditions, conditions which I divide into four groups.

1. The mathematics teacher must have a certain *general education.* We cannot have teachers with good knowledge of mathematics but who are outside that field rather ignorant. He should know one or two foreign languages, history, social science, politics, at least one art, being prepared to share the interests of his pupils outside mathematics whatever they are.

It should be possible to give this general education in (*high*) *school,* but at the university the student must have an open mind and be willing to widen his area of knowledge in these and other fields, but I would not have special courses in general education at the university.

Only one exception: If these courses are not given in the secondary school, I would in the first university year have a course in philosophy, especially in theory of knowledge, and one in general psychology which also could give insight in methods of learning and studying at a university.

2. The next two groups should cover *what* to teach and *how* to teach this material. We may also use the names: *academic* and *pedagogical* training. Both are necessary. First it is absolutely necessary that the mathematics teacher knows not only the material that he presents to his own students, but that he besides has knowledge going deeper.

A well known Swede, representing for years mathematics in the Board of Education, said the other day, when he retired from the Board: "A teacher must not only be the best in his class, he must be souverain in the knowledge material in the textbook". He must be able to answer questions from his students which go "higher up", but he must also know the foundations of his subject, the structures; he must know the working methods of mathematics, have a sure knowledge of such elements as definitions, postulates, theorems, problems, and of deduction and proofs, etc.

We therefore understand why the academic courses must have a very large part of the training, especially since mathematics has a solid position in the schools in mostly all countries.

First, it is quite clear that the amount of „what" will change from school level to school level.

Another question is not so clear. In most countries the academic courses are given at universities, in others we find special institutions for future teachers. (Poland).

Relatively fewer and fewer of the university students of mathematics in these days go to be teachers. The teachers may have some specific needs. It is therefore understandable that some countries have formed special training institutions for future teachers. But I think people going into research or industry more or less have the same needs as future teachers, they also need pure *mathematics*, and besides, I do not think it wise to force the students when they are as young as 18 to make a choice between school teaching and another mathematical career.

It is very important that the academic courses are given to students by mathematicians who have done research work and are real scientists, and if possible also have been school teachers and know the problems of the classroom. This was often the case in the old days when many university professors (like Weierstrass) started their career in a secondary school.

One solution would be to have scientific representation of the subject matter supplemented by practical comments by an experienced and capable school teacher, indicating the best way of using the material in a classroom (as an example, introduction of positive and negative numbers or of complex numbers).

Some courses may still be left being in between the academic and the pedagogical courses. Italy have courses based upon Klein's *Elementarmathematik vom höheren Standpunkt aus* and Enriques' *l'Enciclopedia d'elle mathematiche elementari*. In other countries you find so-called foundations courses, courses giving the background for the curriculum in schools. Such courses could best be given by school teachers who at the same time are mathematicians, but here every country must find its own solution.

In some countries there is a sort of an atomic system for the academic courses with many smaller courses, each ending with an examination (mostly a written one). In other countries we also find smaller courses, but only one, two or three examinations covering a group of such courses.

I think it is irrelevant for our main problems to discuss which of these two systems is best. As a model we may consider a university organization where it is possible to take mathematics courses on different levels. The first level A would correspond to level I for schools; the second level B for level II, and the third level C for level III. I assume that courses on level A and B have examinations which are evaluated, and that to level C there will be attached some longer work which demands *days* (*or months*), that is, writing a sort of a thesis or solving a larger problem or "show the relationship between the results obtained by several authors" (Dutch report).

To get a degree it is necessary in all countries to have taken a certain number of courses and passed the corresponding examinations. These courses can cover mathematics alone or one or more courses in other subjects.

A study of mathematics and nothing else would be too onesided. For level C I would suggest one more subject, one which applies mathematics (physics, biology, sociology, psychology), but I like in this case to give the students complete freedom, and for instance permit mathematics and a foreign language, mathematics and philosophy etc.

On level B with mathematics as a minor, it would be possible to have one major (physics?) or two other minors.

Students taking level A examinations in mathematics would have their main interests in other fields. It is not necessary to discuss how these studies could be organized.

I said before that to each part of the mathematics course in school should correspond a larger course at the university. This may be going too far. If a teacher has a solid course in theory of matrices it must be possible for him to teach theory of determinants without a university course in that field. Important is that the teacher at least has had one complete course in a discipline of mathematics where he can see the whole system with definitions, axioms, theorems, deductions, proofs, etc. But some courses in schools are so specific, so different from others, that an equivalent course at the university is necessary. To be able to teach *calculus* in school you certainly need a calculus course at the university. The same is true for *probability* and *statistics*.

I think I have already mentioned a course which should be compulsory for all the three levels — a course on the structure of mathematics, on the working methods, etc. I have seen an American book with the following chapters which give you some ideas on what I have in mind: Language, Symbols, Compound Statements, Arguments and Proofs, The Axiomatic Method, Introduction to Sets, Logic and Sets, The Structure of Sets, Number Sets, Conditions on Sets, Problem Solving, Relations, Functions, Counting, Probability.

Some of these chapters may be treated later in special courses, but a course of this kind should be a very good introduction to the study of mathematics, and also give the students ideas which they must have if they are to be *good* teachers. Such a foundation course should be taken by all future mathematics teachers, and *all* should further take a course in *history of mathematics.*

For teachers on level A I would add two courses, one in geometry and one in algebra, giving the background for the more elementary teaching in these two fields. It should not be necessary to give detailed plans for these courses. In the algebra course we must have basic concepts of a set, phrases, sentences, equations, inequalities, numbers systems (rational, real, complex), absolute values, truth sets, graphs, etc.; in the geometry course figures defined as set of points, deduction and deductive theory in geometry, measurement, coordinate systems, transformations, geometric intuition, vectors, etc. What is important is to give teachers of mathematics on the first level a sure knowledge and insight in the subject matter to enable them to be good teachers. These courses should not just be a repetition of similar secondary school courses, but should go deeper and higher, and be richer and wider.

For a teacher on the B level, I would have the same three courses which I just mentioned, but also some more. It is not easy to measure such courses. Some students learn fast, others are slow learners. Some students cover two subjects at the same time, other work besides their studies. I dare say that at least one year of thorough study of mathematics is necessary for teachers on the B level, probably more: $1\frac{1}{2}$ year.

These studies should comprise a course in linear algebra-sets, groups, rings, integral domains, fields of real and of complex numbers, linear equations, determinants, and matrices would be some of the chapters. Further, a course in calculus, especially giving the basis ideas of a function, sequences and series, limits, derivation, the two integrals and some simple differential equations. A course in geometry must cover vectors, topology, transformations, foundations, may be some projective geometry and other disciplines.

Besides this, it should be possible to take some elective courses, for instance partial differential equations, probability and statistics, group theory, elementary number theory, numerical analysis, measurement and integral theory, rational mechanics, linear programming, game theory, etc.

A course in statistics and probability must be compulsory for all teachers in the level *C*. Here we also should have a course in mathematical analysis, analytic functions, advanced calculus, differential equations, functions of a complex variable, etc. Further, a course in number theory and may be one in algebra, then studies in a rather large field selected by the student himself from which his special "homework" — thesis, problems, or whatever it may be, is taken. All courses should be strong and thorough, given by real scientists and mathematicians no matter whether at a university or at a special institution for future teachers.

I assume of course that no lecture is just a recitation from a textbook. The lectures should give hints, ideas and impulses. The students should be given the opportunity to make comments and raise questions. The examination papers should not only ask for giving back what is mechanically memorized, but should demand an independent understanding and mastery. During the studies exercises must be given and discussed. Without exercises no effective study!

Today we have several difficulties 1) the shortage of mathematics teachers in schools, 2) the competition from industry, computers, etc. taking the best mathematicians paying them better, 3) the modernized school programs put new demands.

In some countries refresher courses have been introduced in the summer or also in the school year in the evenings or by correspondence.

Professor O. Ore said in a lecture: "A university training which is 20 years old, is too old if it is not supplemented and renewed".

In the U.S.A. films have been prepared or will be made on probability, calculus, designed for mathematics teachers.

A short time ago a clever mathematics teacher said: "Oh, if I could be given permission to go back to the university for half a year".

We have here a very important problem. If we cannot retrain our mathematics teachers, say more than 30 years old, then we can have no reform in mathematics teaching in our schools!

In practically all countries, says the FIPESO-report, teachers in service have felt the need of refresher courses in order to keep pace with recent development in the academic and pedagogical field. Attendance is optional, courses are often organized by the teachers associations.

The conclusion of the FIPESO report is this:

In-service training is a field where close co-operation between teachers and teachers' organizations on one hand and the authorities on the other must be regarded as indispensable in all countries. Let us hope that, in a future not too distant, an efficient in-service training, generously aided by the state, will be looked upon as an essential condition for the improvement of the school system of a country.

In the recommendations it was said:

Teachers attending such courses should receive adequate grants to cover their expenses.

3. The next field is the pedagogical or professional training, the „how" courses. As you know, they have not always been a part of the training of mathematics teachers, and even today this training does not seem to be compulsory in all countries.

I am quite convinced that it should be. Very few are „born" teachers, but most future teachers can profit by proper instructions by professional training, and they can get insight in those many problems in elementary school mathematics where academic wisdom is not sufficient. The students must further study different textbooks (also from other countries), discuss examinations and tests used in mathematics, use of tables and instruments, homework and many other problems which need to be discussed before actual teaching starts. Besides having these socalled methods courses, the students must know the school laws and regulations in the country, educational theory, history of education, educational psychology (learning, youth development, intelligence etc.) and hygiene, general didactics, elements of sociology, comparative education, audiovisual aids, programmed instruction. Further, they should observe teaching in school classes and teach there, being in the end inspected and given a mark, grade or paper indicating their teaching ability.

Instruction of this kind is mostly given in special institutions which also can be parts of a university. The teachers in these institutions must of course have experience from schools; they must be or must have been school teachers, with a strong background in their subject field.

The courses we find have different length — one semester, one year or more. We find a two-year course where professional instruction is combined with teaching. It is a trial period for which the young teacher is paid. In this case the academic training is supposed to be finished earlier. In other cases the academic and the professional training go parallel, which may be wise. On the other hand, I consider it best for the schools to get teachers - even student teachers— who by an academic degree can show that they really know mathematics.

Normally a one year professional or pedagogical course should be sufficient. If the duration is only one semester, all young teachers ought to be given advisors, elder teachers who can assist them, and also if necessary, together with the principal and/or an inspector evaluate their work in school and thereby, if wanted, correct the first mark of teaching ability.

The pedagogical institutions will certainly be developed in the future and will be getting more tasks. They should not only train future teachers but also take up research work, try to find the best teaching methods, evaluate programmed instruction etc. Try to find out why some children have difficulties with mathematics, try to find out how children learn mathematical concepts, at what age they understand a proof and many other problems where psychology and mathematics must cooperate. It must be possible to construct a much better organized and more effective mathematics teaching than we have today.

4. As previously stated, there are four necessary conditions for being a good teacher. The last and fourth group is more difficult to define. We have here personal attributes and qualities which cannot easily be provoked or improved by instruction or teaching. What I have in mind is this: The good teacher must *understand* the students, must be able to follow them, be in contact with them, and must be *open* and *free*. We have here for a great part natural gifts, but I have seen students change during their professional course, and I think more could be done. We need mathematics teachers who are open, free and understanding, who are not afraid of the textbook nor of their students.

To sum up, what I have said is this:

1. Good mathematics teachers must not only know their subject matter, but also have a wider general background in sciences as well as in humanities and in relationships in daily life.
2. They must be open and free, understand their students and their problems in mathematics and to a certain degree also outside mathematics.

3. To many mathematics teachers need a *wider area of knowledge*, more adapted to the programs in school, and they need a *deeper-going knowledge*. This should be given by mathematicians who not only give informative knowledge, but who at the same time inspire their students to active, independent work and creative invention, give them courage to use their imagination, and, on the whole, teach them in the *same way which these students later when they are teachers shall use* in their own class rooms.
4. The professional, pedagogical training must be compulsory in all countries. It can be improved, especially in many cases by discussing more than it is being done at present, all problems which may arise in mathematics classes in schools. The pedagogical institutions ought to start research work by experiments which could show the best teaching methods, and by taking up other problems in mathematics instruction.

The future mathematics teachers must be given sufficient time for student teaching (observations as well as teaching) and should in his first two years as a teacher be given assistance and guidance by elder, experienced teachers. First after these two years, a final mark or grade for his teaching ability should be given.

The first five to ten years may give us trouble: we have too few teachers and we must retrain elder teachers so that they can be able to teach the new programs. But I am optimistic — I say that in at most ten years we shall have a sufficient number of well prepared, good mathematics teachers. But it will be necessary to have refresher courses for teachers in service. Let us remember that in Denmark now 50 percent of *all* mathematics teachers have taken a 14 day summer course in modern mathematics. It must be a task for ICMI to find how such courses can be organized in the most effective way.

VAN DE REDACTIE

In het verslag van het Internationaal Mathematisch Congres te Stockholm-1962, dat werd afgedrukt in het decembernummer van deze jaargang[1]), is aangekondigd dat de teksten van de drie grote voordrachten gehouden voor de sectie VIII (Education) in Euclides zouden worden afgedrukt. Het leek de redactie juist de voordrachten in één nummer te brengen. Daarom zijn de april- en meinummers tot één dubbelnummer samengevoegd. De redactie beveelt deze belangrijke voordrachten die u hiervoor vindt, in uw ernstige belangstelling aan. Het volgende nummer van Euclides verschijnt 1 juni.

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[1]) Euclides, **38**, p. 100 sq.

CURSUSSEN MODERNE WISKUNDE VOOR LERAREN

Op 5 maart 1963 werd door Zijne Excellentie de Staatssecretaris van Onderwijs, Kunsten en Wetenschappen het volgende schrijven (VHMO-212210[1]) verzonden:

Aan de rectoren van de openbare lycea en de directeuren
van de rijkshogereburgerscholen[1])

Zoals U bekend zal zijn heeft mijn ambtsvoorganger in 1961 een Commissie ingesteld waaraan de opdracht is verleend de modernisering van het onderwijs in de wiskunde bij het v.h.m.o. in studie te nemen en mij te adviseren inzake de maatregelen die hiertoe zouden moeten worden getroffen. Deze Commissie heeft een interimrapport ingediend waarin zij — als eerste maatregel — voorstelt cursussen te organiseren waarin de leraren in de wiskunde de gelegenheid wordt geboden kennis te nemen van de moderne ontwikkeling van de wiskunde. Ik heb de Commissie verzocht zich te willen belasten met de organisatie en deel u dienaangaande het volgende mede.

De cursussen zullen worden georganiseerd in Utrecht, Groningen en Eindhoven. De eerste reeks zal worden gehouden in Utrecht en Groningen van 23 tot en met 28 september 1963, te Eindhoven van 16 tot en met 21 september 1963. Des morgens geven hoogleraren colleges. In aansluiting daaraan werken de deelnemers des middags onder leiding van wetenschappelijke ambtenaren aan vraagstukken.

Deze eerste reeks zal in januari 1964, vermoedelijk in aansluiting aan de kerstvacantie, worden vervolgd met een tweede, op dezelfde voet in te richten. Het is de bedoeling dat de deelnemers van de eerste cursus ook de tweede volgen.

Teneinde zo nodig leiding te kunnen geven aan de studie zullen in de periode tussen beide reeksen cursussen regionale bijeenkomsten worden georganiseerd, waarop wetenschappelijke ambtenaren de problemen die zich bij de studie hebben voorgedaan zullen bespreken. Aan deze cursussen kunnen slechts bevoegde leraren deelnemen. De deelneming is kosteloos. Reis- en verblijfkosten komen voor Rijksrekening.

Ik acht deze cursussen van groot belang voor het onderwijs. Ik verzoek U dan ook leraren, die willen deelnemen, hiertoe in de gelegenheid te stellen. Ik keur goed dat U voor het volgen van de cursussen buitengewoon verlof verleent.

De Commissie heeft mij medegedeeld dat zij het voornemen heeft in het begin van dit jaar te Utrecht, Groningen en Eindhoven in een bijeenkomst voor de leraren in de wiskunde de aard en de bedoeling van deze cursussen uiteen te zetten. Ik verwijs U hiervoor naar de hierbij gaande circulaire van de Commissie.

Ik verzoek U de inhoud van deze brief en de bijgaande circulaire ter kennis te brengen van de aan Uw school verbonden leraren in de wiskunde.

De staatssecretaris van
onderwijs, kunsten en wetenschappen,
w.g. (prof. dr. H. H. Janssen)

De in deze brief genoemde circulaire van de Commissie Modernisering leerplan wiskunde, eveneens gedateerd 5 maart 1963, luidde:

Blijkens zijn schrijven van heden, no. 212210[1], hoofdafdeling V.H.M.O., heeft de Staatssecretaris van Onderwijs, Kunsten en Wetenschappen, de Commissie modernisering leerplan wiskunde belast met de organisatie van van Rijkswege te geven cursussen voor leraren in de wiskunde waarin deze leraren de gelegenheid zal worden geboden, kennis te nemen van de moderne ontwikkeling van de wiskunde. Enige gegevens aangaande deze cursussen heeft de Staatssecretaris U in zijn voornoemde brief ter kennis gebracht. De cursussen worden georganiseerd door de Commissie met medewerking van een Commissie van Bijstand, bestaande uit vertegenwoordigers van de drie Pedagogische Centra, de Verenigingen Liwenagel en Wimecos en de Redactie van het tijdschrift Euclides. Gezien het bijzonder karakter van deze cursussen heeft de Commissie gemeend dat een mondelinge toelichting inzake de aard

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[1]) Afschriften van deze brief werden gezonden aan gemeentebesturen, besturen van bijzondere scholen en aan de rectoren en directeuren van gemeentelijke en bijzondere scholen.

en de bedoeling van deze maatregel zeer wenselijk is. Daarom organiseert de Commissie de volgende bijeenkomsten voor de leraren in de wiskunde:

1. te Utrecht op 22 maart 1963 te 15.00 uur in de aula van de universiteit, Domplein;
2. te Groningen op 29 maart 1963 te 15.00 uur in het Academiegebouw (zaal 306), Broerstraat;
3. te Eindhoven op 26 maart 1963 te 15.00 uur in het voormalig gymnasium Augustinianum, Kanaalstraat.

Op deze bijeenkomsten zullen verschillende sprekers, onder wie een hoogleraar in de wiskunde, het woord voeren. Wegens het grote belang van de cursussen voor de toekomstige ontwikkeling van het onderwijs in de wiskunde, meent de Commissie te mogen aannemen dat voor deze bijeenkomsten genoegzaam belangstelling zal bestaan.

De door de leraren voor het bijwonen van deze bijeenkomsten gemaakte reiskosten kunnen bij het secretariaat van de Commissie worden gedeclareerd.

Plaats en tijdstip van de cursussen zijn U door de Staatssecretaris medegedeeld. Aanmeldingsformulieren voor deelneming zijn verkrijgbaar bij het secretariaat van de Commissie, Boothstraat 17 te Utrecht. Zij moeten vóór 1 mei 1963 aldaar worden ingezonden.

De Voorzitter van de Commissie
modernisering leerplan wiskunde
namens deze,
de secretaris,
w.g. (dr. A. F. Monna)

Bij het samenstellen van dit nummer van Euclides liggen de voorlichtingsbijeenkomsten nog voor ons. Als dit nummer verschijnt zijn ze achter de rug. We kunnen slechts de hoop uitspreken, dat velen van de gelegenheid gebruik gemaakt hebben om zich over deze zo belangrijke zaak te laten inlichten.

Het is beslist noodzakelijk, dat wij, leraren van het VHMO, kennisnemen van de moderne ontwikkeling van de wiskunde, dat wij niet alleen ervan weten, maar dat wij werkelijk doordrongen zijn van het feit, dat onze opleiding, als we die meer dan 10 of 15 jaar geleden voltooiden, volkomen verouderd is. Wil het wiskundeonderwijs in Nederland op de scholen voor VHMO in moderne zin aangepast kunnen worden dan is het noodzakelijk dat er bij de docenten begonnen wordt en wij juichen dan ook het initiatief van de Commissie Modernisering Leerplan Wiskunde en het naar aanleiding daarvan genomen besluit van de Staatssecretaris van harte toe.

Laten wij dus met elkaar tonen dat deze uitstekende wijze van aanpakken van dit urgente probleem onze volledige instemming heeft. Ieder van ons melde zich aan voor de cursussen. Nu ons op zo een aantrekkelijke wijze de kans geboden wordt is het onze plicht hem te grijpen!

AMK

REGELMATIGE ZEVENHOEK EN LEMNISCAAT VAN BERNOULLI

door

J. C. G. NOTTROT
Kol. der Genie b.d.
's-Gravenhage

I. *Inleiding*

1. Sedert september 1955 ontving ik vier deeltjes van de „Aperçu de la théorie des polygones réguliers" van de Brusselse auteur-éditeur Pierre A. L. Anspach. Volgende delen zijn misschien al verschenen of nog op komst.

Dit is een merkwaardig, maar ook een zeer eigenaardig geschrift. Een schat van vondsten misschien is uitgestrooid in een kryptogrammatische tekst en in figuren die door zwier aan duidelijkheid inboeten. Anderzijds bloeit tussen de abrupte, soms haast in code gestelde zinnen, een poëtische devotie voor de schoonheid der Mathematiek.

Wanneer men de speurzin en het geduld heeft van de archeoloog, herontdekt men uit dit werk 's schrijvers vondsten en wordt er als deze door geboeid.

2. Een dezer ontdekkingen is de relatie tussen de regelmatige zevenhoek en de lemniscaat van Bernoulli.

Zoals bekend is de lemniscaat van Bernoulli de meetkundige plaats van de punten, waarvoor *het produkt* der afstanden tot twee polen constant is. De l.v.B. is door deze eigenschap de vierde in het kwartet met de ellips, de cirkel en de hyperbool, voor welke resp. *de som, het quotiënt* en *het verschil* dier afstanden constant is.

Bij de l.v.B. domineren twee maten: de halve poolafstand (tevens de breedte) en de halve lengte. Promoveren wij de eerste tot lengteëenheid in onze verdere beschouwingen, dan is de laatste $= \sqrt{2}$.

Kiest men nu buiten de lemniscaat een punt op afstanden 1 en $\sqrt{2}$ van de twee polen en trekt men om dit punt een cirkel met straal $= 1$, dan zijn drie van de vier snijpunten van deze cirkel met de lemniscaat hoekpunten van een in de cirkel te beschrijven regelmatige zevenhoek. In de bijgaande figuur zijn dit de hoekpunten 2, 10 en 12 van de regelmatige zevenhoek 2-4-6- . . -14-2.

Alvorens deze zeer schone eigenschap, die hoogstwaarschijnlijk door Pierre Anspach het eerst werd ontdekt, te bewijzen en er nog vele andere merkwaardige eigenschappen aan toe te voegen, zal ik met u uit de regelmatige veertienhoek het arsenaal van formules te voorschijn halen, dat ons straks van dienst zal zijn.

Over het gebied dat door Pierre Anspach werd geëxploreerd zal ik, hier en daar in zijn spoor tredend, mijn eigen weg kiezen.

II. De regelmatige veertienhoek

3. Elk hoekpunt van de regelmatige veertienhoek heeft behalve de middellijn, tot de overige hoekpunten nog zes verschillende afstanden. Stel de straal weder = 1 en naar opklimmende lengte de bedoelde zes afstanden: u, x, v, y, w en z. Hiervan zijn x, y en z de overeenkomstige drie afstanden in de regelmatige zevenhoek.

Uit de formule $k = 2 \sin \frac{n}{7} . 90^\circ$ berekent men, voor n achtereenvolgens = 1 t/m 6, de zes koordelengten:

$$u = 0{,}4450 \qquad v = 1{,}2470 \qquad w = 1{,}8020$$
$$x = 0{,}8678 \qquad y = 1{,}5636 \qquad z = 1{,}9499$$

Er zullen weinig waarden zijn, die als dit zestal door zo uitbundig veel eenvoudige betrekkingen onderling zijn verbonden. Dit is evenwel niet uitsluitend het privilege van de regelmatige veertienhoek. In andere regelmatige veelhoeken zal men naar rato eenzelfde overvloed van overeenkomstige betrekkingen kunnen oogsten.

4. Met de stelling van Pythagoras leidt men af:

$$z^2 + u^2 = 4 \qquad w^2 + x^2 = 4 \qquad y^2 + v^2 = 4 \qquad (1)$$

en met de Stelling van Ptolemeus, toegepast op diverse gelijkbenige trapezia met een middellijn tot basis:

$$z^2 - u^2 = 2w \qquad w^2 - x^2 = 2v \qquad y^2 - v^2 = 2u \qquad (2)$$

Uit (1) en (2) volgt:

$$\begin{matrix} z^2 = 2 + w & w^2 = 2 + v & y^2 = 2 + u \\ u^2 = 2 - w & x^2 = 2 - v & v^2 = 2 - u \end{matrix} \qquad (3)$$

Uit paren gelijkvormige gelijkbenige driehoeken, resp. met het toppunt in het middelpunt of in een der hoekpunten op de cirkel, vindt men:

$$uz = x \qquad xw = y \qquad vy = z \qquad (4)$$

en uit het produkt dezer formules:

$$\boldsymbol{uvw = 1} \qquad (5)$$

Uit de Stelling van Ptolemeus, toegepast op gelijkbenige trapezia met evenwijdige zijden x, y of z, resp. u, v of w, komen voort:

$$
\begin{aligned}
xy &= v^2 - u^2 = w - u \qquad & vw &= y^2 - u^2 = w + u \\
xz &= y^2 - x^2 = v + u \qquad & uw &= v^2 - x^2 = v - u \\
yz &= w^2 - u^2 = w + v \qquad & uv &= x^2 - u^2 = w - v
\end{aligned} \tag{6}
$$

De gelijkheid van het 2de en 3de lid dezer formules volgt uit (3). Het rechtse stel is ingevolge (5) resp. gelijk aan $\frac{1}{u}$, $\frac{1}{v}$ en $\frac{1}{w}$, en dus is:

$$u(w+u) = v(v-u) = w(w-v) = 1 \tag{7}$$

Uit (3), (4) en (6) zijn nog af te leiden:

$$
\begin{aligned}
ux &= z - y \qquad & wz &= z + y \\
uy &= y - x \qquad & vz &= y + x \\
wy &= z + x \qquad & vx &= z - x
\end{aligned} \tag{8}
$$

alsmede: $x^2y^2z^2 = (2-v)(2+u)(2+w) = 7$
dus

$$\mathbf{xyz = \sqrt{7}} \tag{9}$$

Ten slotte nog:

$$xyz = z + y - x \tag{10}$$

$$uvw = w - v + u \tag{11}$$

(10) en (11) zijn dus ook resp. $= \sqrt{7}$ en $= 1$.

$$u^2 + v^2 + w^2 = 5 \tag{12}$$

$$x^2 + y^2 + z^2 = 7 \tag{13}$$

III. Bewijs voor de betrekking tussen de regelmatige zevenhoek en de lemniscaat van Bernoulli

5. In de bijgaande figuur zijn op de cirkel om O de met 1 t/m 14 genummerde hoekpunten van een regelmatige veertienhoek aangegeven. Uit 11 zijn koorden getrokken naar 6, 2 en 14. Hun lengten, gemeten met de straal als eenheid, zijn dus w, w en v.

Koorde 2-11 werd met een afstand $= u$ verlengd tot F_2. Het produkt van F_2-11 met F_2-2 is dus $u(w+u)$ en dit is volgens (7) gelijk aan 1. Hieruit volgt, dat voor elke rechte lijn door F_2 die de cirkel snijdt, het produkt der in de straal als eenheid gemeten afstanden van F_2 tot de twee snijpunten gelijk aan 1 is.

F_2-7 is gelijk en evenwijdig aan 11-6, want 6-7 is gelijk en evenwijdig aan 11-F_2. F_2-7 gaat dus ook door hoekpunt 10, F_2-7 is w en F_2-10 is $w-v$. Eveneens volgens (7) is het produkt $w(w-v)$ gelijk aan 1.

Een overeenkomstige redenering geldt voor F_2-13 ten opzichte van 11-14. Het produkt van F_2-13 en F_2-12 is $v(v-u)$ en ook dit is volgens (7) gelijk aan 1.

Voor F_2B, de snijlijn door O, is insgelijks F_2A maal F_2B gelijk aan 1. Hieruit volgt:

$$\mathbf{F_2O = \sqrt{2}}$$

Koorde 1-9 (of F_1-9) staat loodrecht op F_2-7. In de rechthoekige driehoek F_1CF_2 ligt het punt 10 op $\frac{1}{2}(v - u)$ van C, dus is $CF_2 = w - \frac{1}{2}(v + u) = w - \frac{1}{2}xz$ en $CF_1 = \frac{1}{2}(y + z) = \frac{1}{2}wz$. Bijgevolg is het kwadraat van de hypotenusa F_1F_2 gelijk aan

$$\begin{aligned}(w - \tfrac{1}{2}xz)^2 + \tfrac{1}{4}w^2z^2 &= \\ w^2 - wxz + \tfrac{1}{4}z^2(w^2 + x^2) &= \\ w^2 + z^2 - wxz &= \\ 4 + v + w - wxz &= \\ 4 + yz - yz &= \mathbf{4}\end{aligned}$$

F_1F_2 is dus van gelijke lengte als de middellijn van de cirkel O.

Voor de hoekpunten 2, 10 en 12 zijn de produkten van hun afstanden tot F_1 en F_2, resp. $u(w + u)$, $w(w - v)$ en $v(v - u)$, volgens (7) alle gelijk aan 1. Deze hoekpunten van een regelmatige zevenhoek liggen dus op de in de figuur getekende lemniscaat van Bernoulli met polen F_1 en F_2.

IV. Merkwaardigheden van de „heptal”.

6. Een heptal (Gr.: hepta = zeven) is een driehoek, waarvan de hoeken zich verhouden als 1 : 2 : 4. De kleinste hoek is dus gelijk aan de middelpuntshoek van de regelmatige veertienhoek. In de figuur is driehoek 2-10-12 zulk een heptal. De zijden van de heptal, wederom gemeten met de straal van de omgeschreven cirkel als lengteëenheid, zijn x, y en z, de afstanden van de hoekpunten tot F_1 en F_2 zijn resp. u, w en v en $\frac{1}{u}$, $\frac{1}{w}$ en $\frac{1}{v}$. De afstanden van O tot de zijden x, y en z zijn resp. $\frac{1}{2}w$, $\frac{1}{2}v$ en $\frac{1}{2}y$.

Het oppervlak van de heptal is dus

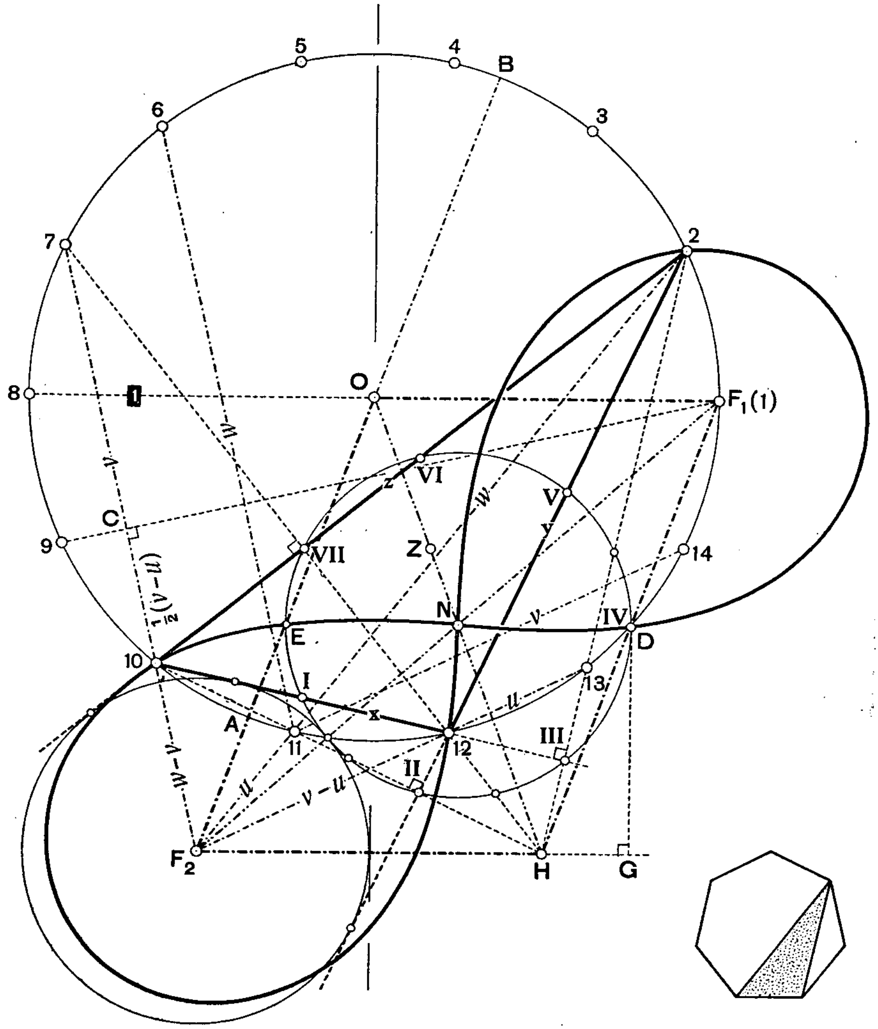
$$\tfrac{1}{4}(wx + vy - uz) = \tfrac{1}{4}(y + z - x) = \tfrac{1}{4}\,xyz = \tfrac{1}{4}\sqrt{7}$$

7. Uit de figuur blijkt, dat de rechte lijnen die F_2 verbinden met de hoekpunten van de heptal, de bissectrices zijn van de hoek bij 2 en van de buitenhoeken bij 10 en 12. Bijgevolg is F_2 het middelpunt van de aan de zijde x aangeschreven cirkel van de heptal.

Het oppervlak van de heptal is dus ook het produkt van $(y + z - x)$ met de halve straal van deze cirkel. Hieruit volgt dat de straal van de aangeschreven cirkel $= \frac{1}{2}$ is.

8. De zoëven genoemde bissectrices 7-F_2, 2-F_2 en 13-F_2 zijn ten opzichte van de loodrecht op 1-8 staande middellijn symmetrisch gelegen met de hoogtelijnen 2-H, 7-H en 10-H van de heptal. Het hoogtepunt H ligt dus t.o.v. genoemde lijn symmetrisch met F_2. Dientengevolge is OF_1HF_2 een parallellogram met zijden 1 en $\sqrt{2}$ en diagonalen $\sqrt{2}$ en 2.

9. Uit de figuur is gemakkelijk te zien en te bewijzen, dat de zijden x, y en z van de heptal de middelloodlijnen van haar hoogtelijnen zijn, wanneer deze worden gemeten van het hoogtepunt H tot de ándere snijpunten met de cirkel, dus resp. tot de punten 13, 11 en 7.



10. De zijde HF_1 van het parallellogram OF_1HF_2 heeft halverwege een snijpunt D met de cirkel O (want $\frac{1}{2}\sqrt{2} \times \sqrt{2} = 1$). Uit de rechthoekige driehoek DGF_2 is te berekenen dat D evenals O op afstand $\sqrt{2}$ van F_2 ligt. Daar $DF_1 \times DF_2$ nu weer $= 1$ is, is D dus weer tevens een punt van de lemniscaat van Bernoulli, evenals ook het tegenovergelegen punt E, dat halverwege F_2O ligt.

Ten opzichte van de heptal heeft D déze bijzondere ligging, dat zijn afstanden tot haar hoekpunten 12, 2 en 10 resp. $\frac{1}{2}x\sqrt{2}$, $\frac{1}{2}y\sqrt{2}$ en $\frac{1}{2}z\sqrt{2}$ zijn, en tot de middens van de zijden y, z en x, of tot de voetpunten der hoogtelijnen op x, y en z, resp. $\frac{1}{2}x$, $\frac{1}{2}y$ en $\frac{1}{2}z$. Het bewijs voor het eerste berust weer op de Stelling van Ptolemeus, voor het tweede en derde bijvoorbeeld op de formule voor de zwaartelijn in een driehoek.

11. OH is de „as van Euler" van de heptal. Op een derde van O ligt daarop het zwaartepunt Z en op de helft het middelpunt N van de negenpuntcirkel. N is dus ook het midden van F_1F_2 en is dus het knooppunt en centrum van de lemniscaat van Bernoulli.

Zoals bekend, gaat de negenpuntcirkel door: *a*. de middens van de zijden, *b*. de voetpunten van de hoogtelijnen en *c*. de middens van de bovenstukken der hoogtelijnen. Bovendien raakt hij de aangeschreven cirkels.

De negenpuntcirkel snijdt uit de zijden x, y en z, c.q. verlengd, resp. een stuk $\frac{1}{2}y$, $\frac{1}{2}z$ en $\frac{1}{2}x$. De afstand van N tot deze zijden is resp. $\frac{1}{4}v$, $\frac{1}{4}u$ en $\frac{1}{4}w$. Die afstand is telkens het gemiddelde van de afstanden van F_1 en F_2 tot de betrokken zijde. Tot de zijde z bijvoorbeeld is dit $\frac{1}{2}[(\frac{1}{2}v - \frac{1}{2}u) + \frac{1}{2}] = \frac{1}{4}(v - u + 1) = \frac{1}{4}w$.

Uit de in het voorgaande gegeven lengten vindt men met behulp van de Stelling van Pythagoras dat ook de straal van de negenpuntcirkel $= \frac{1}{2}$ is.

Zowel de aangeschreven cirkel als de negenpuntcirkel passen dus tussen de evenwijdige raaklijnen aan de lemniscaat die haar breedte begrenzen.

12. De cirkel van Feuerbach heeft hier aan zijn negen bijzondere punten nog niet genoeg. In de eerste plaats draagt hij het raakpunt met de aangeschreven cirkel, welk punt bovendien op cirkel O ligt (omdat $\frac{1}{2} \times 2 = 1$). En ten tweede gaat hij door D en E, daar deze punten, zoals wij zagen, op afstand $\frac{1}{2}$ van N liggen.

13. Tenslotte als apotheose deze verrassende eigenschap:
Zes van de „ordinaire" punten van deze negenpuntcirkel, namelijk de middens van de zijden en de voetpunten van de hoogtelijnen, vormen te zamen met D **een regelmatige zevenhoek.**

De zijden hiervan zijn $\frac{1}{2}x$ lang. In de figuur zijn de hoekpunten genummerd met I t/m VII.

H en Z zijn resp. de gelijkvormigheidspunten van deze zevenhoek en de twee maal zo grote regelmatige zevenhoeken der oneven en der even punten op cirkel O.

De in dit punt genoemde eigenschappen zijn met het in de voorgaande punten opgestapelde bewijsmateriaal gemakkelijk aan te tonen.

14. De rijkdom aan bijzondere eigenschappen van de combinatie heptal-lemniscaat van Bernoulli is hiermede nog bij lange na niet uitgeput. Trek bijvoorbeeld in de hoekpunten van de heptal de raaklijnen aan de lemniscaat en men verkrijgt een nieuwe heptal met F_1 als middelpunt en F_1 D$(= \frac{1}{2}\sqrt{2})$ als straal van de omgeschreven cirkel. Van deze heptal is O het middelpunt van de aan de kleinste zijde aangeschreven cirkel.

Of trek in de hoekpunten van de heptal de normalen en er ontstaat wederom een heptal, thans met het snijpunt van F_1H en 8-F_2 tot middelpunt van de omgeschreven cirkel (met straal $= \frac{1}{2}\sqrt{14}$) en met O weder tot middelpunt van de aan de kleinste zijde aangeschreven cirkel.

De combinatie heptal-lemniscaat van Bernoulli is voor de zwervers door meetkundig-algebraïsche dreven een eldorado. Zij mogen er de ontdekker Pierre Anspach dankbaar om zijn.

UIT DE OPENINGSTOESPRAAK VAN DE VOORZITTER VAN WIMECOS TOT DE ALGEMENE VERGADERING VAN 28 DECEMBER 1962.

„Het verenigingsjaar heeft als belangrijkste kenmerk een gespannen rust. Er gebeurt veel, t.w. er staat veel op stapel in ons onderwijs, maar we merken er hoegenaamd niets van.

Geruisloos is de mechanica als examenvak van het programma verdwenen. Het afgelopen jaar zijn de laatste examens in dit vak gehouden. Voor menig leraar zal dit een minder aangename gebeurtenis zijn geweest. Het vak is nu getrokken bij de natuurkunde en de docenten in de fysica zijn ervan overtuigd dat het nu in goede handen is. Wij wiskundigen zien echter een deel van het vak in diskrediet geraken. Persoonlijk heb ik jarenlang als wiskundeleraar de mechanica onderwezen en nu ik de laatste jaren ook leerlingen klaarmaak voor het examen in de natuurkunde, voel ik duidelijk, dat het vak mechanica zo minder tot zijn recht komt. Ik ben het niet met diegenen eens, die het vak mechanica als doodgelopen beschouwden.

De verdeling van de uren-erfenis van de mechanica heeft zeer onze waardering. De 6-uren wiskunde in de vierde klasse van de H.B.S.-B, zullen het onderwijs daar zeker ten goede komen. Tegen diegenen, die vinden dat ze dat 6e uur niet nodig hebben zou ik willen zeggen, dat dit alleen dan juist zou zijn indien in de 4e klasse de resultaten van de wiskunde zeer goed waren. Van groot belang is dat deze ruimere tijd wordt besteed aan een eerdere behandeling van de diff. rekening, zodat deze kennis zo spoedig mogelijk kan worden gebruikt bij de andere exacte vakken. Dit zou door de fysici op prijs worden gesteld.

Het vervallen van het vak mechanica impliceert, dat de naam van onze vereniging niet juist meer is. Ook de kosmografie dreigt te vervallen, temeer, daar dit vak onder de naam sterrenkunde in de mammoetwet als niet verplicht vak voorkomt. Het bestuur heeft echter besloten de naam van de vereniging niet te veranderen, deels uit historische overwegingen, deels om redenen van financiële aard. Wel heeft de redactiecommissie van Euclides een voorstel ingediend om het titelblad iets te wijzigen t.w. in: Tijdschrift voor de didactiek der wiskunde. Het ontbreken van de woorden mechanica en kosmografie zal echter geen reden zijn, om artikelen over deze onderwerpen, die een specifiek wiskundig karakter hebben, niet op te nemen.

Vorig jaar heeft mijn voorganger Dr. Wansink enkele mededelingen gedaan over de modernisering van het wiskundeprogramma en de instelling van de desbetreffende staatscommissie. De resultaten en vorderingen van deze commissie zijn geheim, zodat we daar geen mededelingen over kunnen verwachten. Wel zal het, volgens de voorzitter Prof. Leeman, lange tijd duren voor de commissie met haar werk gereed is. De zeer dringende noodzaak van de modernisering — we zijn toch al zo ver achter t.o.v. het buitenland — maakt het noodzakelijk, dat er in vlotter tempo wordt gewerkt. Over de doceerbaarheid van nieuwe onderwerpen behoeft de commissie zich niet al te veel zorgen te maken. Dikwijls lijken onderwerpen didactisch veel moeilijker dan ze blijken te zijn, omdat juist de didactiek zich meestal weet aan te passen. U denkt maar eens aan de problemen waar de natuurkunde voor komt te staan bv. bij de kernfysica en de relativiteitsmechanica.

Van groot belang wordt geacht, dat de docenten goede gelegenheid wordt geboden zich te herbekwamen in de theorie, die betrekking heeft op de moderne onderwerpen. Noodzakelijk is, dat dit beter zal gebeuren dan indertijd bij de statistiek. Men moet niet iemand als cursusleider aanstellen, die in vlot tempo de theorie doorloopt in lezingen op hoog niveau, maar een didacticus, die de leraren weet te boeien en enthousiast te maken voor de nieuwe programma's. Dit kan zeer zeker geschieden door het kiezen van het juiste tempo en het zoeken naar veel oefenstof, die ook onder leiding kan worden gemaakt. Op deze wijze kan het spelelement ook hierbij zijn intrede doen. Men vergete niet, dat het opnemingsvermogen en de geheugencapaciteiten van volwassenen belangrijk minder zijn dan bij onze jeugdige leerlingen.

Een woord van grote waardering verdienen de organisatoren van de wiskundeolympiade. Vele collega's juichen deze instelling van harte toe en zijn zeker bereid hiervoor extra moeite en tijd te geven. Ieder van ons beseft ten volle met welk een pijnlijke nauwgezetheid de opgaven moeten worden samengesteld.

Met spanning zien we de nieuwe eerste ronde tegemoet. We menen, dat de opgaven van deze nieuwe ronde wat de moeilijkheidsgraad betreft meer differentiatie zouden moeten vertonen, omdat voor vele enthousiaste deelnemers de gegeven opgaven alle te moeilijk waren, hetgeen deprimerend heeft gewerkt en dat is nu juist *niet* de bedoeling van de olympiade.

Veel genoegen beleven we nog steeds aan ons leerlingentijdschrift

Pythagoras. Mochten er scholen zijn, waarvan nog geen leerlingen abonnee zijn, dan zou ik de betreffende docenten ten sterkste willen raden een woord van aanbeveling tot de leerlingen te richten. Veel administratie is er niet voor nodig. De beide redacteuren broeder Erich en de heer Krooshof hebben veel succes met hun onvermoeide arbeid.

Ons eigen tijdschrift Euclides heeft, behalve de reeds genoemde wijziging in 't titelblad, nog een verandering ondergaan. De wiskundewerkgroep van de W.V.O. maakt nu deel uit van de redactie. Dr. P. M. van Hiele zal hierin zitting nemen en we verwachten dat ons tijdschrift daardoor nog lezenswaardiger zal worden.

Vele artikelen in Euclides zijn van de hand van de heer Wijdenes. Wij betreuren, dat hij vandaag hier niet aanwezig kan zijn en wel speciaal wegens het feit dat deze nestor van de onderwijswereld enkele dagen geleden 90 jaar is geworden. Wij allen mogen wensen, dat wij op die leeftijd nog over zoveel energie en vitaliteit zullen beschikken als de heer Wijdenes. We hopen hem hier nog dikwijls in ons midden te kunnen begroeten.

Het op 19 april 1962 te Utrecht gehouden Congres van leraren in de wis- en natuurkunde heeft veel belangstelling getrokken. De organisatie lag in handen van onze vereniging, samen met de zusterverenigingen. De vakantiecursus van het mathematisch centrum is dit jaar zo druk bezocht, dat wegens plaatsgebrek zelfs niet aan alle aanvragen kon worden voldaan. Het bijwonen van dit soort samenkomsten is niet alleen belangrijk wegens het gebodene, maar ook wegens het onderlinge contact met vakgenoten. Dit contact is één der beste methoden om ons vak met frisheid en enthousiasme te blijven uitoefenen.

De nieuwe examenregeling t.a.v. de mondelinge examens van H.B.S.-B en Gymnasium-B heeft voor de wiskunde in het algemeen veel waardering gevonden. Hoe men hiertegenover staat bij de andere exacte vakken zullen we nog van onze zusterverenigingen te weten komen.

Daar de toename van ons ledental vorig jaar te gering was, hebben we dit jaar een circulaire doen uitgaan om alle collega's, die nog geen lid zijn van onze vereniging, op te wekken dit te worden. Volgens mededeling van de secretaris is het resultaat, dat er zich dit jaar 117 nieuwe leden hebben aangemeld. Wij hopen dat de nieuwe leden de vereniging een warm hart zullen gaan toedragen en actief zullen deelnemen in de werkzaamheden. Hoe groter ons ledental is, des te groter zeggingskracht zal onze vereniging kunnen hebben in de gang van zaken bij het wiskundeonderwijs."

RECREATIE

Nieuwe opgaven met oplossingen en correspondentie over deze rubriek gelieve men te zenden aan Dr. P. G. J. Vredenduin.

86. Van een dominospel, waarvan de laagste steen dubbel 0 en de hoogste dubbel 7 is, is het aantal stenen een kwadraat, nl. 36. Dit is weer het geval, als de hoogste steen dubbel 48 is; het aantal stenen is dan 35^2. Is dit nog vaker het geval en zo ja, is het oneindig vaak het geval? (B. Kootstra)

87. Geen puzzel, maar een meetkundevraagstuk, dat het karakter van een puzzel heeft. De oplossing moet gevonden worden uitsluitend met behulp van meetkundekennis van een leerling van de eerste klasse. Dus geen gebruik van tafels, geen goniometrie, enz. De opgave luidt:

Van driehoek ABC is gegeven $AC = BC$ en $\angle C = 100^0$. Op het verlengde van AC ligt E zo, dat $AE = AB$. Op het verlengde van BC ligt D zo, dat $AD = CD$. Bereken $\angle EDA$. (Dr. J. T. Groenman)

OPLOSSINGEN

(zie voor de opgaven het vorige nummer)

82. De volgende bijzonder elegante oplossing is ingezonden door collega Drs. C. G. Möhlmann te Hamersveld.

Noemen we het aantal wegingen k, dan moet om 10 lichamen naar hun gewichten te ordenen $2^k \geqq 10!$, waaruit volgt $k \geqq 22$. We hebben dus minimaal 22 wegingen nodig. De volgende methode laat zien, dat men met 22 wegingen steeds volstaan kan.

We noemen de lichamen: a, b, c, d, e, f, g, h, i en j. Na 5 wegingen weten we b.v. a < b, c < d, e < f, g < h, i < j. De vijf zwaarste lichamen b, d, f, h en j zijn nu door 7 wegingen volgens hun gewicht te ordenen (vgl. de oplossing van nr 65. in Euclides **37**, IX). Stel we krijgen b < d < f < h < j. Wegens a < b weten we nu: a < b < d < f < h < j.

Wegens e < f kunnen we nu door 2 wegingen e inordenen, en daarna wegens c < d ook c door 2 wegingen. Daarna kunnen we wegens i < j door 3 wegingen i inordenen, en ten slotte wegens g < h ook g door 3 wegingen. Het inordenen van e, c, i en g geschiedt door het lichaam eerst te vergelijken met het middelste van het drietal resp. zevental (vgl. het begin van de oplossing van nr. 65). In totaal kunnen we dus met 22 wegingen volstaan.

83. Men moet 8 maal een „horizontale" en 8 maal een „verticale" zijde van een klein vierkant volgen. Noemen we een horizontale zijde h en een verticale zijde v, dan zijn er dus 8 letters h en 8 letters v, die op $\dfrac{16!}{8! \times 8!}$ verschillende manieren kunnen worden gerangschikt.

84. De voorwaarde $s(s-a)(s-b)(s-c) = 4s^2$ wordt bij vermenigvuldiging met $\dfrac{8}{s}$ en substitutie van $c - b = x$ en $c + b = y$:

$$(y-a)(a^2-x^2) = 16(a+y) \text{ of}$$

$$a^2 - x^2 = 16\frac{y+a}{y-a} = 16 + \frac{32a}{y-a} = 16 + \frac{32}{\dfrac{y}{a}-1}.$$

Veronderstellen we $a \leqq b \leqq c$, dan is $y \geqq 2a$.

Daar het laatste lid een natuurlijk getal moet zijn, voldoen alleen $\frac{y}{a} = 2, 3, 5, 9,$ 17 en 33, waarbij we vinden $a^2 - x^2 = 48, 32, 24, 20, 18$ en $17.$(1)

Als $y = 2a$, is $x = 0$ en $a^2 = 48$, wat niet voldoet.

Schrijven we $a^2 - x^2 = (a - x)(a + x)$, dan vinden we oplossingen voor a en x door de getallen (1) te schrijven als produkt van twee even of van twee oneven factoren.

De oplossingen zijn:

a	5	6	6	7	9
x	1	2	4	5	7
y	25	18	54	35	27
b	12	8	25	15	10
c	13	10	29	20	17

85. Onderstel het kleinste getal is zestallig geschreven *pqrst*, waarin p, q, r, s en t dus de cijfers van het getal voorstellen. Om te beginnen gaan we dit getal denken als een zeventallig geschreven getal (de cijfers veranderen we daarbij niet). De waarde van het getal wordt daarbij vermeerderd met

$$p(7^4—6^4) + q(7^3—6^3) + r(7^2—6^2) + s.$$

Nu tellen we er op tientallige wijze 1963 bij. Om de gedachte te bepalen zou het resultaat er als volgt kunnen uitzien:

$$\begin{array}{r} 12453 \\ 1963 \\ \hline 14416 \end{array}$$

Er is dus bijgeteld 3, afgetrokken 4 . 7, bijgeteld $2 . 7^3$. (Alleen als het voorlaatste cijfer van het bovenste getal een 0 is, is er een andere mogelijkheid. De lezer kan zonder moeite verifiëren, dat deze mogelijkheid niet tot een resultaat leidt.) Nu moet dus

$$1105p + 127q + 13r + s + 2 . 7^3 — 4 . 7 + 3 = 1963.$$

Waaruit we vinden $p = 1$, $q = 1$, $r = 5$, $s = 5$. Verder kan voor t gekozen worden 0, 1, 2 of 3. Voor $t = 0$ vinden we als uitkomst 3685 en 1722.

BOEKBESPREKING

Synopses for modern secondary school mathematics. 310 p. Uitgave van de „Organisation for European Economic Co-operation”, Office for scientific and technical personnel.

Eind 1959 werd door de Organisatie voor Europese Economische Samenwerking een congres georganiseerd over „New Thinking in school mathematics”, een verslag daarvan staat in Euclides 35 (1959-’60, pag. 218-229). Op dit congres werd besloten een commissie te benoemen met tot taak het opstellen van leergangen naar welker model het onderwijs in de aangesloten landen zou kunnen worden gemoderniseerd.

Het is wel duidelijk, waarom een organisatie voor economische samenwerking zich met het wiskundeonderwijs bemoeit. Het gaat om opvoeren van kwantiteit en kwaliteit van wetenschappelijk en technische werkers en daartoe is modernisering van de schoolwiskunde een belangrijk middel. Tevens heeft men de „samenwerking” op het oog: een zekere uniformiteit in de programma’s zou gelijkwaardigheid van de opleidingen garanderen en het gezamenlijk aanpakken van problemen vergemakkelijken.

Deze commissie heeft een interessant stuk werk geleverd. Als vakken stelt ze zich voor: algebra, meetkunde, waarschijnlijkheidsrekening en statistiek, differentiaal- en integraalrekening. De eerste drie in twee ronden: de eerste voor elf- tot vijftienjarigen, de tweede voor ouderen. De leergangen zijn op zeer verschillende wijze uitgewerkt. Waar het geheel nieuwe stof betreft vormen ze een systematisch geheel, andere beperken zich tot een serie opmerkingen. De algebra voor ronde 1 bijvoorbeeld zou men haast zo als leerboekje kunnen uitgeven, het is een bewonderenswaardig stukje werk. De meetkunde daarentegen vertoont veel minder samenhang. Het viel veel moeilijker daar tot een oordeel te komen.

Intussen doet dat er niet veel toe, het komt mijns inziens op mijn oordeel ook niet aan, ik kan het bij een aankondiging laten. Het gaat hier om leerstofvoorstellen, die door de bevoegde instanties moeten worden beoordeeld; dit boek hoort thuis bij de Staatscommissie modernisering leerplan wiskunde. Het zal daar ook wel al in onderzoek zijn, de voorzitter dezer commissie was gedelegeerd op het genoemde congres. Hij zal naar dit rapport al wel uitgekeken hebben.

Dat neemt niet weg, dat kennisname in bredere kring wenselijk is, vandaar dat ik toch deze aankondiging inzend. De studie van dit boek betekent zeker geen verloren tijd.

Hoe men eraan moet komen, wordt niet vermeld, evenmin als de prijs. De uitvoering is slecht, het boek lijkt me gestencild, er zit een slap papieren bandje om.

Haren (Gron.) J. Koksma

R. J. Legger en G. L. Ludolph: *Hogere wiskunde voor de technicus*, J. B. Wolters, Groningen 1962; 411 blz.; prijs: geb. *f* 17.50.

Het boek is bestemd voor het onderwijs aan Hogere Technische scholen. Een naar de vorm voortreffelijk boek; de theoretische behandeling is niet al te streng, maar in het algemeen wiskundig verantwoord.

Een enkele opmerking: op blz. 39 en 40 had de afleiding voor de reeks voor het getal e wel iets scherper gekund. Zo was het eenvoudig geweest om de bewijzen dat voor de rij $u_n = \left(1 + \frac{1}{n}\right)^n$ geldt $u_{n+1} > u_n$ en dat verder de rij sommeerbaar was. Ook heb ik persoonlijk er bezwaar tegen om uitdrukkingen als 0/0 schijnbaar onbepaald te noemen. 0/0 is te allen tijde onzin en juist hiervan is m.i. met een weinig moeite een oefening in exact uitdrukken te maken. Daartegenover staan weer mooie didactische vondsten; op blz. 142 en 143 wordt heel fraai de grafiek van $f(x)$ vergeleken met die van $f'(x)$; op blz. 166 e.v. wordt een bepaalde integraal werkelijk als limiet van een som uitgerekend en daarna direct bepaald met behulp van een primitieve functie. Het verschil tussen de grote rekenarbeid in het eerste en het direct opschrijven in het tweede geval is buitengewoon aardig. Minder fraai vind ik de behandeling van de integratiemethode door middel van substitutie. Zo zonder meer bij de substitutie $z = 2x$ te verklaren, dat dan $dz = 2dx$ gezet moet worden, is wat al te kort.

Daarentegen is de toelichting van de partiële integratie weer zeer fraai.

Als een HTS-er dit boek met vrucht heeft doorgewerkt, en alle 800 vraagstukken gemaakt, dan heeft hij een behoorlijke hoeveelheid wiskundige vaardigheid verworven. Hij kan dan meepraten over een heleboel praktische bruikbare wiskundige technieken: logaritmisch differentiëren, reeksen van Taylor en McLaurin, integratiemethoden, functies van meer veranderlijken, lineaire differentiaal-vergelijkingen van hogere orde, Fourierreeksen, convergentie-onderzoek van reeksen enz.

P. Bronkhorst